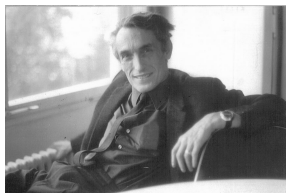


Bijections and Surjections

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Colloque en l'honneur de Marcel-Paul Schützenberger
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Combinatorial Hopf algebras

- Richard Stanley (1972): P -partitions generalizing
 - plane partitions
 - Semistandard Young Tableaux
- Generating series of P -partitions (Schur symmetric functions)
- The function associated to a poset P is *quasi-symmetric* (I. Gessel 1984).
- The set of P -partitions decomposes through linear extensions of P .

- A Hopf-algebraic version of these results
- A generalizations from orders to preorders
- Joint work with Loïc Foissy and Frédéric Patras (2015)

P -partitions

P a labelled poset of degree n . A P -partition is a sequence of nonnegative integers $f = (f(1), \dots, f(n))$ such that:

- ① if $i \leq_P j$, then $f(i) \leq f(j)$.
- ② if $i \leq_P j$ and $i > j$, then $f(i) < f(j)$.

The set of P -partitions is denoted by $Part(P)$.

- if $P = \mathfrak{I}_1^3$, a P -partition is an increasing sequence $(f(1), f(2), f(3))$.
- if $P = \mathfrak{I}_3^1$, a P -partition is a strictly increasing sequence $(f(1), f(2), f(3))$.

Quasi-Symmetric functions

$F \in K[[X_1, X_2, \dots]]$ is quasi-symmetric if in F
coefficient of $X_{i_1}^{a_1} \dots X_{i_n}^{a_n} = \text{coefficient of } X_{j_1}^{a_1} \dots X_{j_n}^{a_n}$
 $\forall \quad i_1 < \dots < i_n, \quad j_1 < \dots < j_n$

QSym: the algebra of quasi-symmetric functions.

A basis of **QSym**: monomial quasi-symmetric functions:

$$M_{(a_1, \dots, a_n)} = \sum_{i_1 < \dots < i_n} X_{i_1}^{a_1} \dots X_{i_n}^{a_n}$$

$$\mathbf{Sym} \subseteq \mathbf{QSym}$$

Multiply means

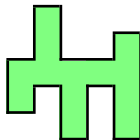
... take 2 objects of the same kind and form a new object of the same kind

Multiply = Assembly

Multiply means

... take 2 objects of the same kind and form a new object of the same kind

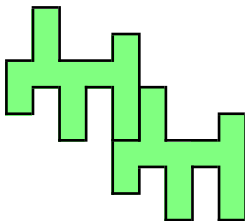
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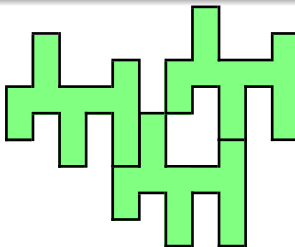
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Comultiply means

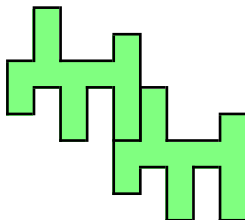
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Comultiply = Decompose

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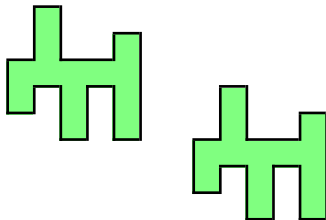
Comultiply = Decompose

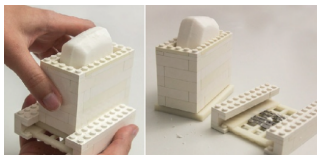


Comultiply means

... take 1 object of one kind and form one or more pairs of objects both of the same kind

Comultiply = Decompose





"ASSEMBLING": Algebra

- A vector space V with a *product* $m: V \otimes V \rightarrow V$;
- a *unity* $1: \diamond \rightarrow V$, where \diamond is a singleton;
- Those applications must verify associativity.

"SPLITTING": Coalgebra

- a *coproduct* $\delta: V \rightarrow V \otimes V$;
- a *counity* $\epsilon: V \rightarrow \diamond$;
- Those applications must verify coassociativity.

"COMPATIBILITY": Bialgebra

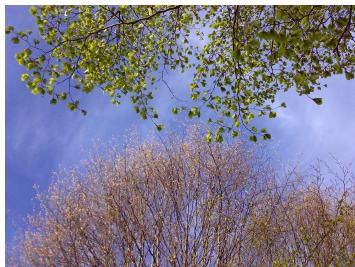
- coproduct and counity must be *algebra morphisms*.

"INVERSION": Hopf Algebra

- there must be an *antipode* $S : V \rightarrow V$
- such that:

$$m \circ (id \otimes S) \circ \delta = 1 \circ \epsilon = m \circ (S \otimes id) \circ \delta$$

- Topology: cohomology of Lie groups and homogeneous spaces
- Combinatorics: general philosophy of Gian Carlo Rota / hereditary structures (William Schmitt)
- Classification and enumeration of discrete structures
- Persi Diaconis: Card shuffling, Hopf algebra of descents
- Theoretical Computer Science: linear logic (Blute, Girard)



- Connes - Kreimer
- Loday - Ronco
- Grossman - Larson

Bijections: Examples, morphisms

Bijections

- Given the finite n -element set $[n] := \{1, 2, \dots, n\}$, a bijection $\sigma : [n] \rightarrow [n]$ is called permutation of $[n]$.
- S_n is the set of permutations of $[n]$.
- We regard the permutation $\sigma \in S_n$ as the list of its successive images:

$$\sigma(1), \sigma(2), \dots, \sigma(n).$$

Bijections

- Given the finite n -element set $[n] := \{1, 2, \dots, n\}$, a bijection $\sigma : [n] \rightarrow [n]$ is called permutation of $[n]$.
- S_n is the set of permutations of $[n]$.
- We regard the permutation $\sigma \in S_n$ as the list of its successive images:

$$\sigma(1), \sigma(2), \dots, \sigma(n).$$

$$\begin{aligned}\sigma &= 53241 \in S_5 & \tau &= 35124 \in S_5 \\ \sigma \circ \tau &= \sigma(\tau(1)), \sigma(\tau(2)), \dots, \sigma(\tau(5)) = 21534\end{aligned}$$

- The vector space KS : linear combinations of all finite permutations:

$$\tau = 53241 + 312 - 1342 \in KS = \bigoplus_{n=0}^{\infty} KS_n$$

- The vector space KS : linear combinations of all finite permutations:

$$\tau = 53241 + 312 - 1342 \in KS = \bigoplus_{n=0}^{\infty} KS_n$$

- The product is *concatenation-type*:

$$\sigma \star \alpha = \sum_{\substack{st(u)=\sigma \\ st(v)=\alpha}} uv.$$

$$\sigma = 213 \in S_3 \quad \alpha = 21 \in S_2 \quad \sigma \star \alpha \in S_5$$

$$213|54 + 214|53 + 215|43 + 324|51 + 325|41 + 435|21.$$

- The coproduct is *unshuffle-type*

$$\Delta(\sigma) = \sum_{i=0}^n \sigma|_{\{1, \dots, i\}} \otimes st(\sigma|_{\{i+1, \dots, n\}})$$

$$3124 \rightarrow \lambda|3124 \quad 1|324 \quad 12|34 \quad 312|4 \quad 3124|\lambda$$

$$\Delta(3124) = \lambda \otimes 3124 + 1 \otimes 213 + 12 \otimes 12 + 312 \otimes 1 + 3124 \otimes \lambda$$

A dual Hopf algebra of permutations

- the product is the *shifted shuffle* $\sigma \star' \alpha = \sigma \sqcup \bar{\alpha}$



A dual Hopf algebra of permutations

- the product is the *shifted shuffle* $\sigma \star' \alpha = \sigma \sqcup \bar{\alpha}$



$$\sigma = 213 \in S_3 \quad \alpha = 21 \in S_2 \quad \bar{\alpha} = 54$$

$$\sigma \star' \alpha = 213 \sqcup 54$$

$$\begin{aligned} &21354 + 21534 + 21543 + 25134 + 25143 + 25413 + \\ &+ 52134 + 52143 + 52413 + 54213 \end{aligned}$$

- The coproduct is *deconcatenation-type*

$$\sigma = 21354$$

$$\lambda|21354 \quad 2|1354 \quad 21|354 \quad 213|54 \quad 2135|4 \quad 21354|\lambda$$

One needs to standardize in order to obtain permutations

$$\Delta'(\sigma) = \lambda \otimes 21354 + 1 \otimes 1243 + 21 \otimes 132 + 213 \otimes 21 + 2134 \otimes 1 + 21354 \otimes \lambda$$

- The antipode T is obtained recursively by the equation

$$m_{\star} \circ (id \otimes T) \circ \Delta = \eta \circ \epsilon = m_{\star} \circ (T \otimes id) \circ \Delta :$$

$$T(\lambda) = \lambda$$

$$T(1) = -1$$

$$T(12) = 21$$

$$T(21) = 12$$

$$T(123) = -321$$

$$T(321) = -123$$

$$T(132) = -312 - 213 + 231$$

$$T(213) = -231 - 132 + 312$$

$$T(231) = -2 \cdot 231 - 132 + 213 + 312$$

$$T(312) = -2 \cdot 312 - 213 + 132 + 231$$

$$T(1234) = -4321$$

$$\vdots$$

Special Posets

A special poset is a partial order \leq_P defined on a totally ordered set, e.g.:

$[n] = \{1, \dots, n\}$, for $n \geq 0$.

We represent it with its *Hasse diagram*.

$$1 = \emptyset; \bullet_1; \bullet_1 \bullet_2, \mathfrak{I}_1^2, \mathfrak{I}_2^1;$$

$$\bullet_1 \bullet_2 \bullet_3, \mathfrak{I}_1^2 \bullet_3, \mathfrak{I}_1^3 \bullet_2, \mathfrak{I}_2^1 \bullet_3, \mathfrak{I}_2^3 \bullet_1, \mathfrak{I}_3^1 \bullet_2,$$

$$\mathfrak{I}_3^2 \bullet_1, {}^2\mathfrak{V}_1^3, {}^1\mathfrak{V}_2^3, {}^1\mathfrak{V}_3^2, {}_2\mathfrak{A}_3^1, {}_1\mathfrak{A}_3^2, {}_1\mathfrak{A}_2^3, \mathfrak{I}_1^3, \mathfrak{I}_1^2, \mathfrak{I}_2^3, \mathfrak{I}_2^1, \mathfrak{I}_3^2, \mathfrak{I}_3^1 \dots$$

Hopf algebra of special posets \mathcal{H}_{SP} (M.-Reutenauer)

- Basis: the set of special posets
- Product: concatenation with shift of labels.

$${}^2V_1^3 \cdot \mathfrak{I}_1^3 = {}^2V_1^3 \mathfrak{I}_4^6, \quad \mathfrak{I}_1^3 \cdot {}^2V_1^3 = \mathfrak{I}_1^3 {}^5V_4^6.$$

- \mathcal{H}_{SP} is freely generated by the set of *indecomposable* special posets:

$$\bullet_1; \mathfrak{I}_1^2, \mathfrak{I}_2^1; \mathfrak{I}_1^3 \bullet_2, \mathfrak{I}_3^1 \bullet_2,$$

$${}^2V_1^3, {}^1V_2^3, {}^1V_3^2, {}_2\Lambda_3^1, {}_1\Lambda_3^2, {}_1\Lambda_2^3, \mathfrak{I}_1^3, \mathfrak{I}_1^2, \mathfrak{I}_2^3, \mathfrak{I}_2^1, \mathfrak{I}_3^2, \mathfrak{I}_3^1.$$

Hopf algebra of special posets \mathcal{H}_{SP} (M.-Reutenauer)

- P a special poset of degree n .
 $I \subseteq [n]$ is an ideal of P :

$$(x \in I, y \in P, x \leq_P y) \implies (y \in I).$$

- Coproduct:

$$\Delta(P) = \sum_{I \text{ ideal of } P} Std(P \setminus I) \otimes Std(I).$$

$$\begin{aligned} \Delta\left({}^2V_1^3\right) &= {}^2V_1^3 \otimes 1 + 1 \otimes {}^2V_1^3 + !_1^3 \otimes ._2 + !_1^2 \otimes ._3 + ._1 \otimes ._2 ._3 \\ &= {}^2V_1^3 \otimes 1 + 1 \otimes {}^2V_1^3 + !_1^2 \otimes ._1 + !_1^2 \otimes ._1 + ._1 \otimes ._1 ._1 \end{aligned}$$

- Vector space: **QSym** with monomial quasi-symmetric functions as basis.
- product: quasi-shuffle (stuffle = STicky shUFFLE) (cfr. Hoffmann, Multiple Zeta Values)

$$M_{(a)}M_{(b)} = M_{(a,b)} + M_{(b,a)} + M_{(a+b)},$$

$$M_{(a,b)}M_{(c)} = M_{(a,b,c)} + M_{(a,c,b)} + M_{(c,a,b)} + M_{(a,b+c)} + M_{(a+c,b)}.$$

- Coproduct: deconcatenation

$$\Delta(M_{(a_1, \dots, a_n)}) = \sum_{i=0}^n M_{(a_1, \dots, a_i)} \otimes M_{(a_{i+1}, \dots, a_n)}.$$

Linear extensions of special posets

P a special poset of degree n . A linear extension of P is a total order \leq_{tot} on $[n]$ such that:

$$(i \leq_P j) \implies (i \leq_{tot} j).$$

The set of linear extensions of P is denoted with $L(P)$.

$$L\left(\begin{smallmatrix} 1 \\ \vdots \\ 3 \end{smallmatrix} \vee_3 \begin{smallmatrix} 2 \\ \vdots \\ 3 \end{smallmatrix}\right) = \left\{ \begin{smallmatrix} 1 \\ \vdots \\ 3 \end{smallmatrix}, \begin{smallmatrix} 2 \\ \vdots \\ 3 \end{smallmatrix} \right\} = (321) + (312)$$

$$L\left(\begin{smallmatrix} 4 \\ \vdots \\ 3 \end{smallmatrix} \vee_1 \begin{smallmatrix} 2 \\ \vdots \\ 1 \end{smallmatrix}\right) = \left\{ \begin{smallmatrix} 4 \\ \vdots \\ 3 \\ 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 4 \\ \vdots \\ 3 \\ 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 4 \\ \vdots \\ 2 \\ 3 \\ 1 \end{smallmatrix} \right\} = (1234) + (1324) + (1342)$$

Special poset associated to a permutation

Let $\sigma \in \mathfrak{S}_n$. We define a total order \leq_{tot} on $[n]$ by:

$$(i \leq_{tot} j) \iff (\sigma(i) \leq \sigma(j)).$$

We obtain in this way all special posets such that \leq_P is total.

$$(123) \longleftrightarrow \begin{array}{c} 3 \\ \vdots \\ 2 \\ 1 \end{array}$$

$$(132) \longleftrightarrow \begin{array}{c} 2 \\ \vdots \\ 3 \\ 1 \end{array}$$

$$(213) \longleftrightarrow \begin{array}{c} 3 \\ \vdots \\ 1 \\ 2 \end{array}$$

$$(231) \longleftrightarrow \begin{array}{c} 2 \\ \vdots \\ 1 \\ 3 \end{array}$$

$$(312) \longleftrightarrow \begin{array}{c} 1 \\ \vdots \\ 3 \\ 2 \end{array}$$

$$(321) \longleftrightarrow \begin{array}{c} 1 \\ \vdots \\ 2 \\ 3 \end{array}$$

Hence, linear extensions of any special posets can be seen as permutations.

Generating series of P -partitions

Let P a special poset of degree n .

$$\gamma(P) = \sum_{f \text{ } P\text{-partition of } P} X_{f(1)} \cdots X_{f(n)}.$$

It is a quasi-symmetric function.

$$\begin{aligned} \gamma \left(\begin{smallmatrix} 3 \\ \vdots \\ 2 \\ 1 \end{smallmatrix} \right) &= \sum_a X_a^3 + \sum_{a < b} X_a X_b^2 + \sum_{a < b} X_a^2 X_b + \sum_{a < b < c} X_a X_b X_c \\ &= M_{(3)} + M_{(1,2)} + M_{(2,1)} + M_{(1,1,1)} \end{aligned}$$

$$\begin{aligned} \gamma \left(\begin{smallmatrix} 1 \\ \vdots \\ 2 \\ 3 \end{smallmatrix} \right) &= \sum_{a < b < c} X_a X_b X_c \\ &= M_{(1,1,1)} \end{aligned}$$

Stanley's theorem

Let P be a special poset.

$$① \quad \text{Part}(P) = \bigsqcup_{Q \in L(P)} \text{Part}(Q).$$

$$② \quad \gamma(P) = \sum_{Q \in L(P)} \gamma(Q).$$

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Theorem

The map $\gamma : \mathcal{H}_{SP} \longrightarrow \mathbf{QSym}$ is a surjective morphism of Hopf algebras.

Theorem

- The following map is a surjective Hopf algebras morphism:

$$L : \begin{cases} \mathcal{H}_{SP} & \longrightarrow KS \\ P & \longrightarrow \sum_{\sigma \in L(P)} \sigma \end{cases}$$

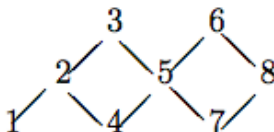
- The following is a commutative diagram of Hopf algebra morphisms:

$$\begin{array}{ccc} \mathcal{H}_{SP} & \xrightarrow{L} & KS \\ & \searrow \gamma & \downarrow \phi \\ & & \mathbf{QSym} \end{array}$$

- Given a skew partition $\lambda/\mu = 443/21$:
- label the cases of its Ferrers diagram in the reading order

1	2	3	
:	4	5	6
:	:	7	8

- turn it counterclockwise to obtain $P = P_{\lambda/\mu}$:



- the $P_{\lambda/\mu}$ -partitions are the semi-standard Young tableaux.

- Given a skew partition λ/μ :

$$P = P_{\lambda/\mu} \Rightarrow \gamma(P_{\lambda/\mu}) = s_{\lambda/\mu} \in \mathbf{Sym}$$

is not only quasi-symmetric, but a symmetric function (the symmetric Schur function associated to the partition λ/μ).

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- Is the reciprocal true?

$$\gamma(P) \in \mathbf{Sym} \Rightarrow \exists \lambda/\mu : P = P_{\lambda/\mu}?$$

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- Is the reciprocal true?

$$\gamma(P) \in \mathbf{Sym} \Rightarrow \exists \lambda/\mu : P = P_{\lambda/\mu}?$$

A conjecture of Stanley (1972)

If the generating function of a special poset is symmetric, then the poset is isomorphic to a convex subposet of the discrete plane.

\mathcal{H}_{SP}

QSym

KS

P

$\gamma(P)$

$L(P)$

$P_{\lambda/\mu}$

symmetric

plactic-closed

\mathcal{H}_{SP}

QSym

KS

P

$\gamma(P)$

$L(P)$

$P_{\lambda/\mu}$

symmetric

plactic-closed

Theorem (CM 1993)

$L(P)$ is plactic-closed $\Leftrightarrow \exists \lambda/\mu : P = P_{\lambda/\mu}$.

Surjections, finite topologies

Surjections

- A map $f : [n] \rightarrow [k]$ can be represented as a word, the list of its successive images:

$$f(1), f(2), \dots, f(n).$$

- If f is surjective, then

$$\{f(1), \dots, f(n)\} = [k].$$

- Ordered partitions $(A_1, A_2, \dots, A_k) \leftrightarrow$ surjections f via $A_i = \{j : f(j) = i\} = f^{-1}(i).$

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$$f = (3132212) \rightarrow (\{2, 6\}, \{4, 5, 7\}, \{1, 3\}).$$

Let P a special poset of degree n . We put:

$$\Gamma(P) = \sum_{f \text{ } P\text{-partition of } P} X_{f(1)} \dots X_{f(n)} \in K\langle\langle X_1, X_2, \dots \rangle\rangle.$$

Hopf algebra of packed words **WQSym** (Thibon-Novelli)

- A packed word is a word $f(1) \dots f(n)$ such that $\{f(1), \dots, f(n)\} = [k]$ for a certain k .

$$1 = \emptyset; (1); (12), (21), (11);$$

$$(123), (132), (213), (231), (312), (321),$$

$$(122), (212), (221), (112), (121), (211), (111) \dots$$

Hopf algebra of packed words **WQSym** (Thibon-Novelli)

- For any packed word $f(1) \dots f(n)$, we put:

$$M_{f(1)\dots f(n)} = \sum_{\text{Pack}(g(1)\dots g(n))=f(1)\dots f(n)} X_{g(1)} \dots X_{g(n)},$$

a basis of a subalgebra **WQSym** of $K\langle\langle X_1, X_2, \dots \rangle\rangle$.

- The canonical projection $K\langle\langle X_1, X_2, \dots \rangle\rangle \rightarrow K[[X_1, X_2, \dots,]]$ induces a surjective algebra π morphism **WQSym** \rightarrow **QSym**.

$$\pi(M_{f(1)\dots f(n)}) = M_{(\#f^{-1}(1), \dots, \#f^{-1}(\max(f)))}.$$

$$\pi(M_{(2,2,1,3,1,2,2)}) = M_{2,4,1}.$$

Hopf algebra of packed words **WQSym** (Thibon-Novelli)

- Product: If $\max(f) = k$ and $\max(g) = l$,

$$f.g = \sum_{\alpha \in QSh(k,l)} \alpha \circ (f \otimes g).$$

$$\begin{aligned}(132).(21) = & (13254) + (14253) + (15243) + (14352) \\ & + (15342) + (15432) + (24351) + (25341) \\ & + (25431) + (35421) + (13221) + (13231) \\ & + (13241) + (132|32) + (13242) + (14231) \\ & + (14232) + (14243) + (14321) + (14332) \\ & + (14342) + (24321) + (24231) + (24341)\end{aligned}$$

Hopf algebra of packed words **WQSym**

- Coproduct: if $\max(f) = n$,

$$\Delta(f) = \sum_{k=0}^n f_{|[k]} \otimes \text{Std}(f_{|\mathbb{N}_{>0} \setminus [k]}).$$

$$\begin{aligned}\Delta((511423)) &= 1 \otimes (511423) + (511423) \otimes 1 + (11) \otimes (5423) \\ &\quad + (112) \otimes (543) + (1123) \otimes (54) + (11423) \otimes (5) \\ &= 1 \otimes (511423) + (511423) \otimes 1 + (11) \otimes (4312) \\ &\quad + (112) \otimes (321) + (1123) \otimes (21) + (11423) \otimes (1).\end{aligned}$$

Theorem

- The following map is a surjective Hopf algebra morphism:

$$\Gamma : \begin{cases} \mathcal{H}_{SP} & \longrightarrow \mathbf{WQSym} \\ P & \longrightarrow \sum_{f \text{ } P\text{-partition of } P} X_{f(1)} \cdots X_{f(n)}. \end{cases}$$

- There exists a commutative diagram of Hopf algebra morphisms:

$$\begin{array}{ccccc} \mathcal{H}_{SP} & \xrightarrow{L} & KS & & \\ & \searrow \Gamma & \downarrow \varphi & \searrow \phi & \\ & & \mathbf{WQSym} & \xrightarrow{\pi} & \mathbf{QSym} \end{array}$$

$$\Gamma\left(\begin{smallmatrix} 1 \\ \vdots \\ 3 \\ 2 \end{smallmatrix}\right) = \sum_{a < b} x_b x_a^2 + \sum_{a < b < c} x_c x_a x_b$$

$$= (211) + (312)$$

$$\Gamma\left(\begin{smallmatrix} 1 \\ \vdots \\ 3 \\ 2 \end{smallmatrix}\right) = \sum_{a < b} x_b^2 x_a + \sum_{a < b < c} x_b x_c x_a + \sum_{a < b < c} x_c x_b x_a$$

$$= (221) + (231) + (321)$$

$$\Gamma\left(\begin{smallmatrix} 2 \\ \vdots \\ 3 \\ 1 \end{smallmatrix}\right) = \sum_a x_a^3 + \sum_{a < b} x_a^2 x_b + \sum_{a < b} x_a x_b x_a + \sum_{a < b} x_a x_b^2$$

$$+ \sum_{a < b < c} x_a x_b x_c + \sum_{a < b < c} x_a x_c x_b$$

$$= (111) + (112) + (121) + (122) + (123) + (132)$$

Preorders

- A preorder on a set X is a transitive and reflexive relation \leq_T .
- If \leq_T is a preorder on X , then:
 - 1 The relation \sim_T defined by:

$$(x \sim_T y) \iff (x \leq_T y \text{ and } y \leq_T x)$$

is an equivalence on X .

- 2 X / \sim_T is partially ordered by:

$$(\bar{x} \leq_T \bar{y}) \iff (x \leq_T y).$$

We represent preorders on $[n]$ by the Hasse diagram of X/\sim .

$$1 = \emptyset ; \bullet_1 ; \bullet_1 \bullet_2, \mathfrak{I}_1^2, \mathfrak{I}_2^1, \bullet_{1,2} ;$$

$$\bullet_1 \bullet_2 \bullet_3, \mathfrak{I}_1^2 \bullet_3, \mathfrak{I}_1^3 \bullet_2, \mathfrak{I}_2^1 \bullet_3, \mathfrak{I}_2^3 \bullet_1, \mathfrak{I}_3^1 \bullet_2,$$

$$\mathfrak{I}_3^2 \bullet_1, \mathfrak{V}_1^3, \mathfrak{V}_2^3, \mathfrak{V}_3^2, \mathfrak{A}_3^1, \mathfrak{A}_3^2, \mathfrak{A}_2^3, \mathfrak{I}_1^3, \mathfrak{I}_1^2, \mathfrak{I}_2^3, \mathfrak{I}_2^1, \mathfrak{I}_3^2, \mathfrak{I}_3^1,$$

$$\bullet_{1,2} \bullet_3, \bullet_{1,3} \bullet_2, \bullet_{2,3} \bullet_1, \mathfrak{I}_{1,2}^3, \mathfrak{I}_{1,3}^2, \mathfrak{I}_{2,3}^1, \mathfrak{I}_3^{1,2}, \mathfrak{I}_2^{1,3}, \mathfrak{I}_1^{2,3}, \bullet_{1,2,3}.$$

Alexandrov's theorem

For any $n \geq 0$, there exists a bijection between the set of preorders on $[n]$ and the set of topologies on $[n]$.

- if \leq is a preorder on X , the set of ideals of \leq defines a topology on X .
- If T is a topology on X , the following relation is a preorder on X :

$$(x \leq y) \iff (\text{every open set of } T \text{ containing } x \text{ contains } y).$$

Hopf algebra of topologies \mathcal{H}_T

- Basis: the set of all finite topologies on sets $[n]$.
- Product: shifted concatenation.

$${}^2V_1^3 \cdot \mathbb{I}_1^3 = {}^2V_1^3 \mathbb{I}_4^6, \quad \mathbb{I}_1^3 \cdot {}^2V_1^3 = \mathbb{I}_1^3 {}^5V_4^6.$$

- \mathcal{H}_T is freely generated by the set of indecomposable topologies:

$$\begin{aligned} & \bullet_1; \mathbb{I}_1^2, \mathbb{I}_2^1, \bullet_{1,2}; \mathbb{I}_1^3 \bullet_2, \textcolor{red}{\mathbb{I}_3^1} \bullet_2, \\ & {}^2V_1^3, {}^1V_2^3, {}^1V_3^2, {}_2\Lambda_3^1, {}_1\Lambda_3^2, {}_1\Lambda_2^3, \mathbb{I}_1^3, \mathbb{I}_1^2, \mathbb{I}_2^3, \mathbb{I}_2^1, \mathbb{I}_3^2, \mathbb{I}_3^1, \\ & \bullet_{1,3} \bullet_2, \mathbb{I}_{1,3}^2, \mathbb{I}_2^{1,3}, \bullet_{1,2,3}. \end{aligned}$$

Hopf algebra of topologies \mathcal{H}_T

Coproduct:

$$\Delta(P) = \sum_{I \text{ open of } P} Std(P \setminus I) \otimes Std(I).$$

$$\begin{aligned} \Delta\left({}^2V_1^3\right) &= {}^2V_1^3 \otimes 1 + 1 \otimes {}^2V_1^3 + \bullet_2 \otimes !_1^3 + \bullet_3 \otimes !_1^2 + \bullet_2 \bullet_3 \otimes \bullet_1 \\ &= {}^2V_1^3 \otimes 1 + 1 \otimes {}^2V_1^3 + \bullet_1 \otimes !_1^2 + \bullet_1 \otimes !_1^2 + \bullet_1 \bullet_2 \otimes \bullet_1 \end{aligned}$$

T -partitions

Let T be a topology on $[n]$. The associated preorder is denoted by \leq_T . A T -partition of T is a sequence $(f(1), \dots, f(n))$ such that:

- $(i \leq_T j) \implies (f(i) \leq f(j))$.
- $(i \leq_T j, \text{ not } i \geq_T j \text{ and } i > j) \implies (f(i) < f(j))$.
- $(i < j < k, i \sim_T k \text{ and } f(i) = f(j)) \implies (i \sim_T j)$.

T -partitions of ${}^1\mathcal{V}_{2,4}^5 \cdot_3$ are sequences $(f(1), \dots, f(5))$ such that:

- $f(2) = f(4) \leq f(1), f(5)$
- $f(2) = f(4) < f(1)$
- $f(3) \neq f(2), f(4)$

If $a < b < c < d$:

$(babaa), (babab), (babac), (bacaa), (bacab), (bacac), (bacad),$
 $(badac), (cabaa), (cabab), (cabac), (cabad), (cacab), (cadab),$
 $(cbabb), (cbabc), (cbabd), (dabac), (dacab), (dbabc).$

Theorem

The following map is a surjective Hopf algebra morphism:

$$\Gamma : \begin{cases} \mathcal{H}_T & \longrightarrow \mathbf{WQSym} \\ T & \longrightarrow \sum_{f \text{ } T\text{-partition of } T} X_{f(1)} \cdots X_{f(n)}. \end{cases}$$

$$\begin{aligned} \Gamma \left(\begin{smallmatrix} 1 & \bullet & 5 \\ & \searrow & \nearrow \\ & 2,4 & \\ & \bullet & \\ & \nearrow & \searrow \\ & 3 & \end{smallmatrix} \right) &= (21211) + (21212) + (21213) + (21311) \\ &\quad + (21312) + (21313) + (21314) + (21413) \\ &\quad + (31211) + (31212) + (31213) + (31214) \\ &\quad + (31312) + (31412) + (32122) + (32123) \\ &\quad + (32124) + (41213) + (41312) + (42123) \end{aligned}$$

Theorem

The following map is a surjective Hopf algebra morphism:

$$\Gamma : \begin{cases} \mathcal{H}_T & \longrightarrow \mathbf{WQSym} \\ T & \longrightarrow \sum_{f \text{ } T\text{-partition of } T} X_{f(1)} \cdots X_{f(n)}. \end{cases}$$

$$\begin{aligned} \Gamma \left(\begin{array}{c} 1 \bullet \\ \vdots \\ 2,4 \\ \vdots \\ 5 \\ \vdots \\ 3 \end{array} \right) &= (21211) + (21212) + (21213) + (21311) \\ &\quad + (21312) + (21313) + (21314) + (21413) \\ &\quad + (31211) + (31212) + (31213) + (31214) \\ &\quad + (31312) + (31412) + (32122) + (32123) \\ &\quad + (32124) + (41213) + (41312) + (42123) \end{aligned}$$

Linear extensions

Let T be a topology on n . A linear extension of T is an ordered partition $A = (A_1, \dots, A_k)$ of $[n]$ such that:

- the equivalence classes of \sim_T are A_1, \dots, A_k ;
- if, in the poset $[n]/\sim_T$, $A_i \leq_T A_j$, then $i \leq j$.

Linear extensions $A = (A_1, \dots, A_k)$ are represented by packed words $(f(1), \dots, f(n))$.

Linear extensions of ${}^1\mathbb{V}_{2,4}^5 \cdot_3$: packed words $(f(1), \dots, f(5))$ such that:

- $f(2) = f(4)$, and $f(1), f(2), f(3), f(5)$ are all distinct.
- $f(2) < f(1), f(5)$.

$(21314), (21413), (31214), (31412),$
 $(32124), (41213), (41312), (42123)$

where $(21314) \rightarrow (\{2, 4\}, \{1\}, \{3\}, \{5\})$.

Another product on **WQSym**

- Second product: If $\max(f) = k$ and $\max(g) = l$,

$$f \sqcup g = \sum_{\alpha \in Sh(k,l)} \alpha \circ (f \otimes g).$$

$$\begin{aligned} (132) \sqcup (21) &= (13254) + (14253) + (15243) \\ &\quad + (14352) + (15342) + (15432) + (24351) \\ &\quad + (25341) + (25431) + (35421) \end{aligned}$$

Theorem

- The following map is a surjective Hopf algebra morphism:

$$L : \begin{cases} \mathcal{H}_T & \longrightarrow (\mathbf{WQSym}, \sqcup) \\ T & \longrightarrow \sum_{f \in L(T)} f \end{cases}$$

- There exists a commutative diagram of Hopf algebra morphisms:

$$\begin{array}{ccc} \mathcal{H}_T & \xrightarrow{\quad L \quad} & (\mathbf{WQSym}, \sqcup) \\ & \searrow \Gamma & \downarrow \varphi \\ & & \mathbf{WQSym} \end{array}$$

Maria Montessori

Who, as an educator, manages to arouse an interest that prompts to choose an action and to pursue it with all one's strength, with an energetic enthusiasm, wakes up the man.

(Chi in educazione è riuscito a suscitare un interesse che porta a scegliere un'azione e ad eseguirla con tutte le forze, con entusiasmo fattivo - ha svegliato l'uomo.)

