

## **Topics:**

- 1) Threshold Networks.
- 2) Some applications: bootstrap percolation, Hopfield model, Regulation Networks, Schelling's Segregation Model ...
- 3) Updating schemes and dynamics over undirected graphs.
- 4) Characterization of the convergence to fixed points or cycles.
- 5) Related decision problems and computational complexity.

## **Threshold networks** $x \in \{0,1\}^n$

$$x'_{i} = H(\sum_{j=1}^{n} w_{ij} x_{j} - b_{i}) \text{ for } 1 \le i \le n$$

$$W = (W_{ij})$$
 the weight integral matrix

$$b = (b_i)$$
 the threshold vector

$$H(u) = 1 \quad \text{if} \qquad u \ge 0$$

## The dynamics

## **Block- sequential updates:**

Consider a partition  $\{I_1, ..., I_p\}$  of the set  $\{1, ..., n\}$ 

We update the blocks one by one:

To update the k-th block we consider the new state of every sites belong to previous blocks.

Parallel or synchronous update: only one block. Every site is updated at the same time.

Sequential update: n-blocks of cardinality one: sites are updated one by one in a prescribed order.

We consider only symmetric integral threshold networks. i.e. W being a symmetric matrix with integral entries.

W=W(G) is the symmetric incidence matrix of a weighted graph G=(V,E)



#### Example of dynamics for symmetric threshold networks

We consider a 4x4 lattice with periodic conditions, nearest interactions, states 0 or 1, and the local majority function:

If the number of ones is bigger or equal to the number of zeros then the site takes the value 1

$$x'_{ij} = 1$$
 iff  $x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1} \ge 2$ 

#### Dynamics: two cycles and fixed points; different behavior for different updates



Some applications.

#### Hopfield Threshold Networks

p vectors to be memorized 
$$\{\varepsilon^1, ..., \varepsilon^p\}$$
  $\varepsilon^k \in \{-1, 1\}^n$ 

The matrix weight:

$$w_{ij} = \frac{1}{p} \sum_{k=1}^{p} \varepsilon_i^k \varepsilon_j^k \qquad i, j \in \{1, ..., n\}$$

Thresholds = 0

#### W is symmetric

#### Dynamics: sequential or asynchronous update.

J. J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities", Proceedings of the National Academy of Sciences of the USA, vol. 79 no. 8 pp. 2554–2558, April 1982

#### Arabidopsis regulation threshold network



Bioinformatics. 1999 Jul-Aug;15(7-8):593-606.

Genetic control of flower morphogenesis in Arabidopsis thaliana: a logical analysis. Mendoza L, Thieffry D, Alvarez-Buylla ER. Demongeot J, G. E, Morvan M, Noual M, Sené S (2010) Attraction Basins as Gauges of Robustness against Boundary Conditions in Biological Complex Systems. PLoS ONE 5(8): e11793. doi:10.1371



Bull Math Biol (2013) 75, 939-966

## The Schelling segregation model Thomas C. Schelling (1969)

Two dimensional lattice with Moore's neighborhood, states {-1,1}

• An individual is unhappy if there are more than k individuals on the other state in its neighborhood

At each step, one lists the unhappy individuals of both species, and then randomly one exchanges two individuals of opposite value.



N. Goles-Domic, E.G., S. Rica, Dynamics and Complexity Of the Schelling segregation model, Phys. Rev E, vol1E83 Pp1-13,2011



## **Bootstrap Percolation**

Given a finite undirected graph G=(V,E)

and an initial configuration of 0's and 1's

Consider the strict majority function operating at each node

What is the relationship between the graph and the proportions of 1's such that updated in parallel every node will become 1?

E. G., P. Montealegre-Barba, I. Todinca, The complexity of the bootstraping percolation and related problems, Theoretical Comp. Science, to appear (2013).

![](_page_14_Figure_0.jpeg)

 $V_i = \{ j \in V / (i, j) \in E \}$ 

![](_page_15_Picture_0.jpeg)

## Attractors on threshold networks over undirected graphs

For arbitrary matrices W previous model may accept, Iterated in parallel or sequentially, long period cycles and transients ....

But when W is symmetric the network converges to fixed point or two periodic cycles (parallel update),

## And, if diag(W)≥0 to fixed point (sequential update).

E.G, J. Olivos, Periodic behaviour of generalized threshold functions, Discrete mathematics, vol 30, pp 187-189, 1980.

E.G., Fixed Point behavior of threshold functions on a finite set, SIAM Journal on Alg. And Discrete Methods, vol 3(4), pp 2554-2558, 1982.

#### Further for W symmetric the network admits an energy:

Parallel update:

$$E(x(t)) = -\sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} w_{ij} x_j(t-1) + \sum_{i=1}^{n} b_i(x_i(t) + x_i(t-1))$$

If diag (W)  $\geq$  0, Sequential update:

$$E(x) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i$$

Which implies that:

1) for the parallel updating the attractors are only Fixed points or two cycles.

2) For the sequential updating and diag(W) $\geq 0$  there are only fixed points.

3) In both situations transients are bounded by  $\alpha ||W||x||b||$ 

 $x' \neq x$ 

 $\Delta E = E(x(t)) - E(x(t-1) < 0 \text{ If and only if } x(t) \neq x(t-2)$ 

And for the sequential iteration

 $\Delta E = E(x') - E(x) < 0 \quad \text{iff} \quad x' \neq x$ 

## The most general dynamical result:

Consider the block-sequential scheme  $s = \{I_1, ..., I_p\}$ 

The symmetrical threshold network *T*=(W, b, *s*)

Let  $W(I_k)$  the sub-matrix associated to the k-th block

If for every  $k \in \{1,...,p\}$   $W(I_k)$  is non-negative-definite The network converges to fixed points

E. G., F. Fogelman-Soulie, D. Pellegrin, Decreasing energy functions as a tool For studying threshold networks, Discrete Applied Mathematics, vol 12, pp261-277, 1985.

## Sketch of the proof:

The update of the k-th block:

$$x' = (x_{I_1}, \dots, x_{I_{k-1}}, x'_{I_k}, x_{I_{k+1}}, \dots, x_{I_p})$$

$$\Delta E = -\sum_{i \in I_k} (x_i' - x_i) (\sum_{j=1}^n w_{ij} x_j - b_i) - \frac{1}{2} \sum_{i \in I_k} (x_i' - x_i) \sum_{i \in I_k} (x_j' - x_j)^{T}$$

$$\Delta E = \sum_{i \in I_k} \delta_i - \frac{1}{2} y^i W(I_k) y$$
where  $y = (x' - x) \in \{-1, 0, 1\}^n$ 

$$\delta_i = -(x_i' - x_i) (\sum_{j=1}^n w_{ij} x_j - b_i)$$

$$x' \neq x \Rightarrow \text{ there exists } i \in \{1, ..., n\} \text{ such that } \delta_i \leq -\frac{1}{2}$$
(since W is an integral matrix)

Then  $\Delta E < 0$ 

We will suppose now that every matrix is the incidence matrix of an undirected graph G=(V,E), so their entries belong to the set {0,1} W=W(G)=  $(w_{ii})$  eventually with loops  $(w_{ii} = 1)$ 

Consider the quantity:

$$\alpha(G) = -n - k + 2m - 4p$$

- n = |V|, m = |E|, (without loops) K = the number of loops, P = the minimum number of edges to remove
  - such that the sub-graph is bipartite.

## Example

![](_page_23_Figure_1.jpeg)

V  = 4	k = 2
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|E| = 6 p = 2

Maximum bipartite sub-graph

 $\alpha(G) = -4 - 2 + 2 \times 6 - 4 \times 2 = -2 < 0$ 

## Theorem-1

Consider an undirected graph G=(V,E), W=W(G), b a threshold vector.

and the network updated in parallel, N= (W, b, {1, ...,n})

 $\alpha(G') < 0$  For any G' sub-graph of G (by deleting vertices)  $\Rightarrow$ 

 $\alpha(G') \ge 0 \implies \text{There exists a threshold vector} \\ \text{such that two cycles appears}$ 

Fixed points for any threshold vector

#### Parallel update

![](_page_25_Figure_1.jpeg)

$$f_{1}(x) = H(x_{2} + x_{3} + x_{4} - \frac{3}{2})$$

$$f_{2}(x) = H(x_{1} + x_{3} - \frac{1}{2})$$

$$f_{3}(x) = H(x_{1} + x_{2} + x_{4} - \frac{3}{2}) \implies (x_{1}, x_{2}, x_{3}, x_{4})$$

$$f_{4}(x) = H(x_{1} + x_{3} + x_{4} - \frac{3}{2})$$

$$(x_1, x_2, x_3, x_4) = (1, 0, 1, 0) \iff (0, 1, 0, 1)$$

There exists a sub-graph with  $\alpha(G) \ge 0$ 

 $\alpha(G) = -2$ 

![](_page_25_Figure_7.jpeg)

 $\alpha(G) = 0$ (2) (1)  $f_1(x) = H(x_2 - \frac{1}{2})$  $f_2(x) = H(x_1 - \frac{1}{2})$  $(1,0) \Leftrightarrow (0,1)$ Two-cycle

## Parallel updating on two families of graphs

![](_page_26_Figure_1.jpeg)

#### Complete graphs with n loops

In this situation, the minimum number of edges to remove to obtain a bipartite graph

$$p = 2q(q-1) \quad \text{for n=2q}$$

$$p = 2q^2 \quad \text{for n=2q+1}$$

$$\alpha(K_n) < 0 \quad \Rightarrow \quad \text{Complete graphs updated in}$$
Parallel converges to fixed points

#### Parallel Updating

n=4	Fixed points	Two-Cycles
0-0-0-0	3≤k≤4	0≤k≤2
	1≤k≤4	k=0
88	Ø	0≤k≤4
88	3≤k≤4	0≤k≤2
220	1≤k≤4	k=0

k=number of loops

![](_page_28_Figure_0.jpeg)

Connected graphs for n=5 with 5 loops.

 $\frac{\alpha(G)}{2} = -n + m - 2p$ 

In red the edges to be removed for a maximum bipartite graphs Theorem-II: attractors for every block-sequential update.

Consider the block-sequential scheme  $s = \{I_1, ..., I_p\}$ 

The symmetrical threshold network *T*=(W, b, *s*)

Let  $G(I_k)$  the graph associated to the k-th block

 $\forall k \in \{1,...,p\} \ \alpha(G') < 0 \ \forall G' \subseteq G(I_k) \Rightarrow \text{fixed points}$ 

 $\exists k \in \{1,...,p\} \text{ and } G' \subseteq G(I_k) \text{ such that } \alpha(G') \ge 0 \Rightarrow \text{ cycles}$ 

## Corollary

Consider an undirected graph G=(V,E) with every loop and the the block-sequential scheme  $s = \{I_1, ..., I_p\}$ 

![](_page_30_Figure_2.jpeg)

Otherwise, there exist graphs and threshold vectors such that cycles appear

Sketch of the proof:

Partition size =1 directly from the fact that diag(W)≥0

Partition size = 2

![](_page_31_Figure_3.jpeg)

 $\alpha(G) = -2$ 

Partition size= 3  $\alpha(G) = -4$ 

## Cycles for block-sequential updates Every undirected graph with at least two connected vertices without loops admits cycles

![](_page_32_Figure_1.jpeg)

$$f_{1}(x) = H(x_{2} + \sum_{j \in V_{1} \setminus \{2\}} x_{j} - \frac{1}{2})$$
  
$$f_{2}(x) = H(x_{1} + \sum_{j \in V_{2} \setminus \{1\}} x_{j} - \frac{1}{2})$$
  
Every site {3, ...,n}  
is constant at state 0

 $\alpha(G(\{1,2\},\{(1,2)\})) = -2 + 2 \times 1 = 0$ 

$$(x_1, x_2, \vec{x}) = (1, 0, \vec{0}) \iff (0, 1, \vec{0})$$
  
Two cycle for any partition  $\tau = \{\{1, 2\}, I_2, ..., I_p\}$ 

## **Non-Polynomial Cycles**

![](_page_33_Figure_1.jpeg)

#### staircase

![](_page_33_Figure_3.jpeg)

Local majority at each vertex

$$f_{3}(x) = H(x_{2} + x_{3'} + x_{4} - \frac{3}{2})$$
$$f_{3'}(x) = H(x_{2'} + x_{3} + x_{4'} - \frac{3}{2})$$

![](_page_34_Figure_0.jpeg)

**Block-Sequential updating** 

$$\tau = \{\{1,1'\},\{n,n'\},\{n-1,(n-1)'\},...,\{3,3'\},\{2,2'\}\}$$

![](_page_35_Figure_2.jpeg)

Union of the first I prime number's staircases of size

$$p_1 + 1 = 3; p_2 + 1 = 4; p_3 + 1 = 6, p_4 + 1 = 8, ..., p_l + 1$$

![](_page_36_Figure_2.jpeg)

Same arguments can be done for the transient time.

#### Computational Complexity of some threshold networks

The class P: problems which me can solve in a serial computer in polynomial time

The class NC: problems which can be solved in a parallel machine (say a PRAM) in Poly-logarithmic time by using a polynomial number of processors

A candidate to be intrinsically serial is to compute the truth value of a circuit (CVP): we Have to do that layer by layer ..... Without surprise CVP is P-Complete.. It is also not difficult to prove that the monotone (only AND and OR gates) circuit problem remains P-Complete.

## **Bootstrap Percolation**

Given a finite undirected graph G=(V,E)

and an initial configuration of 0's and 1's

Consider the strict majority function operating at each node

What is the relationship between the graph and the proportions of 1's such that iterated in parallel every node will become 1?

![](_page_39_Figure_0.jpeg)

 $V_i = \{ j \in V / (i, j) \in E \}$ 

## **Decision problem**

PRE: given an initial configuration and a specific node at value 0. Does there exist T>0 such that this node becomes 1?

Theorem (P. Montealegre, I. Todinca, E.G (2011))

Given a family of undirected graph G:

If the maximum degree  $\geq$  5, PER is P complete.

Else (Maximum degree ≤4) PER belongs to NC

Clearly PER belongs to P, because in almost O(n) steps the dynamics reaches the steady state.

The proof of P-Completeness consist to simulate monotone circuits inside the strict majority dynamics.

![](_page_43_Picture_0.jpeg)

![](_page_44_Figure_0.jpeg)

## Information only flows to the right

![](_page_45_Picture_0.jpeg)

For the case maximun degree  $\leq 4$  one may reduce the problem to compute connected and biconnected components in the graph, which one may do in a PRAM in  $O((\log n)^2)$ 

![](_page_47_Figure_0.jpeg)

# The Complexity of the majority vote rule for planar graphs

![](_page_48_Figure_1.jpeg)

We consider the similar decision problem PRE

This problem has been studied by C. Moore for d-dimensional regular lattices with nearest interactions

![](_page_49_Picture_2.jpeg)

![](_page_49_Figure_3.jpeg)

Von Neumann neighborhood in 2D Nearest neighborhood In 3D

PRE is P-Complete for d ≥ 3 open for d = 2 (C. Moore)

## For planar graphs PRE is P-Complete

(P. Montealegre-Barba, E:G, 2012)

## PRE is in P

Majority is a particular case of a threshold network:

$$F(x) = H(Wx - b)$$

Since G is undirected W is a nxn symmetric matrix and the threshold:

 $b_i = \frac{1}{2} | V_i |$  Odd neighborhood  $b_i = \frac{1}{2} (|V_i| + 1)$  Even neighborhood

The parallel dynamic is driven by

$$E(x(t)) = -\sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} w_{ij} x_j(t-1) + \sum_{i=1}^{n} b_i(x_i(t) + x_i(t-1))$$

Which is strictly decreasing and bounded  $o(n^2)$ 

#### So PER is in P

![](_page_52_Figure_0.jpeg)

![](_page_53_Figure_0.jpeg)

## AND-gate

![](_page_53_Picture_2.jpeg)

## OR-gate

![](_page_54_Figure_0.jpeg)

![](_page_55_Figure_0.jpeg)

Consider a decision problem slightly different than PRE taking Into account the updating scheme over majority functions:

PRE(S): given an initial configuration and a specific node at value 0 and an updating scheme S.
Does there exist T>0 such that this vertex becomes 1 when the updating scheme S is applied?

PRE(S) is NP-Hard for block-sequential updating schemes. (E.G, P. Montealegre-Barba,2013)

This result is a direct consequence from the fact that block- sequential schemes on the majority admit non-polynomial cycles.

![](_page_58_Figure_0.jpeg)

 $q_k = p_k + 1$  The k-th prime number

At every step we will simulating a different true assignment of the variables

There will be 3 layers in the network: the firs are the gadgets simulating variables,

The second: we simulate every clause by joining three different variables with a node which simulates the OR function.

The third layer: we joint every OR to a vertex simulating the AND function.

![](_page_60_Figure_0.jpeg)

Gray=1; White=0

 $p_i = 3$ 

 $x_i = 1 \Leftrightarrow t = ap_i$  $\overline{x}_i = 1 \Leftrightarrow t \neq ap_i$ 

Each variable is 1 for any multiple of the gadget's period

![](_page_61_Figure_0.jpeg)

Updating Scheme.

every two partition in the first layer.
 Every couple of variables.
 Every "clause"
 The AND vertex .

## Recently we proved that The block-sequential majority Prediction is also PSPACE-COMPLETE

**PSPACE-completeness of majority automata networks** 

E. G., Pedro Montealegre , Ville Salo, Ilkka Törmä Theoretical Computer Science 609 (2016) 118–128.

### Summary

PRE for bootstrap percolation: P-Complete for degree ≥ 5. Belongs to NC for degree ≤ 4. In a 2d lattice belongs to NC

For the majority function: PRE is P-complete even for planar graphs

For the Majority Function PRE(S) is NP-HARD

Still open: complexity for a 2D lattice

![](_page_65_Picture_0.jpeg)

![](_page_65_Picture_1.jpeg)