Schützenberger's Star-Free Theorem and what Followed

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March 22, 2016

Schützenberger's Theorem

Regular Languages

Important Reminder: Regular Languages

Finite words over a finite alphabet A:

- Word = finite concatenations of letters in A.
  (Example: "*abba*" and "*bbaabba*" for $A = \{a, b\}$).
- Language = set of words. The language of all words is A^*.
- Regular language = language than can be defined by
  a finite automaton or a regular expression.

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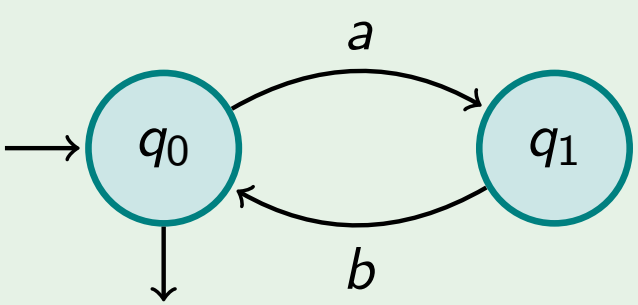
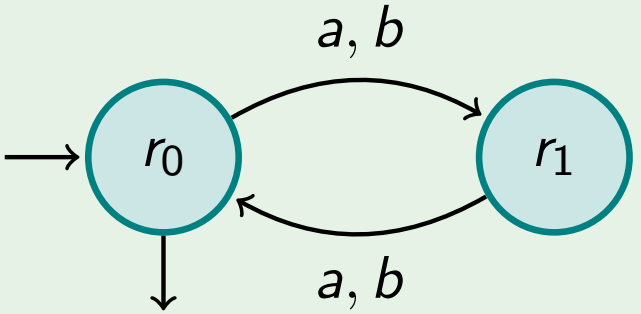


Language $(ab)^*$



Language of words of even length

Star-free Languages: What are they ?

Definition by Induction

- all **finite and co-finite languages** are star-free
  ($\{a\}$ is star-free, $\{b, bab, aa\}$ is star-free, A^* is star-free,...)

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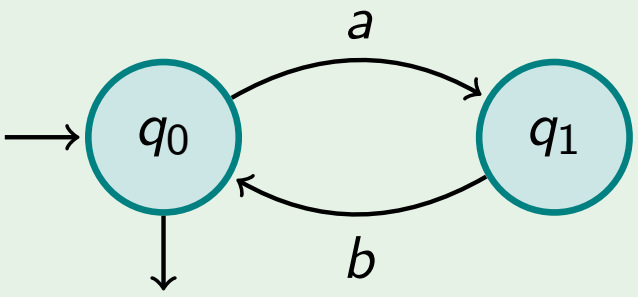
The star-free languages form a **strict** subclass of the regular languages.

Star-free Languages: Examples



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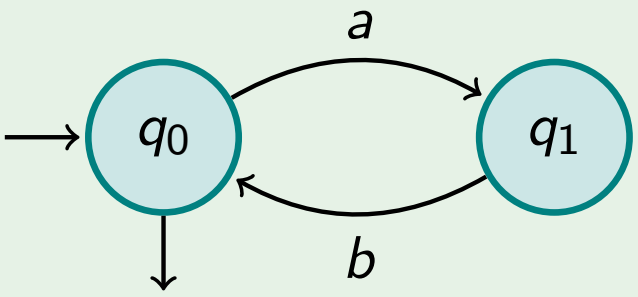


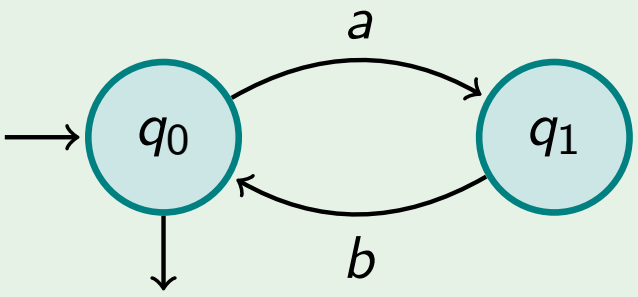
Language $(ab)^*$

This language is star-free

A word is in the language iff

- It does not begin with a "b"
- It does not contain two consecutive "a"
- It does not contain two consecutive "b"
- It does not end with a "a"

Star-free Languages: Examples



Language $(ab)^*$

This language is star-free

A word is in the language iff

- It does not belong to $b \cdot A^*$
- It does not belong to $A^* \cdot aa \cdot A^*$.
- It does not belong to $A^* \cdot bb \cdot A^*$.
- It does not belong to $A^* \cdot a$

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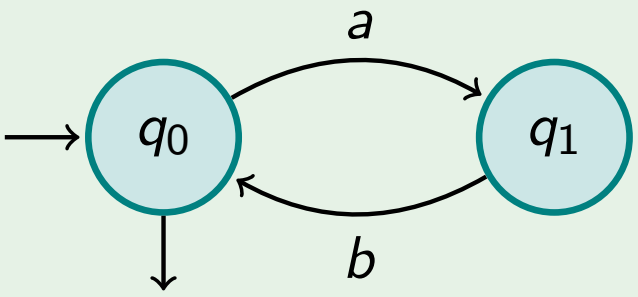

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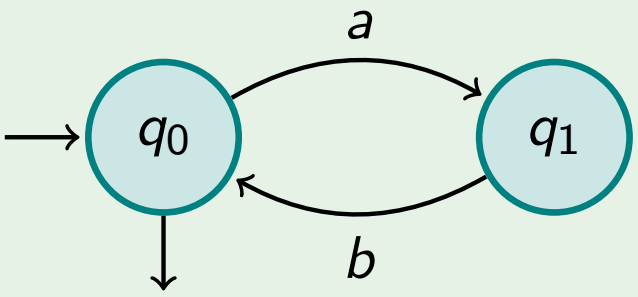

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$$\overline{\overline{b \cdot A^*} \cap \overline{A^* \cdot a}} \cap \overline{\overline{A^* \cdot aa \cdot A^*} \cap \overline{A^* \cdot bb \cdot A^*}}$$

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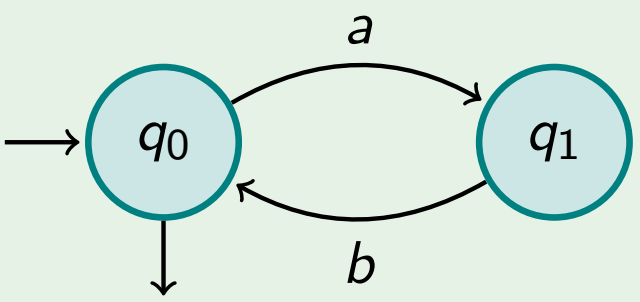

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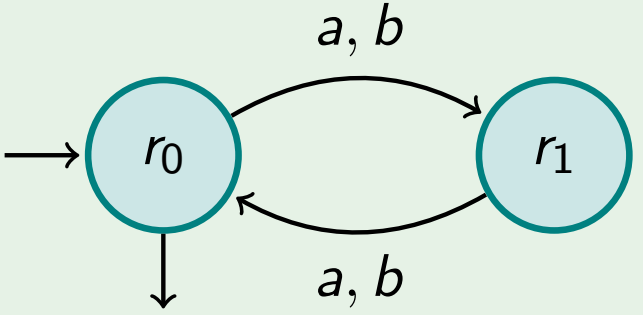

Words of even length

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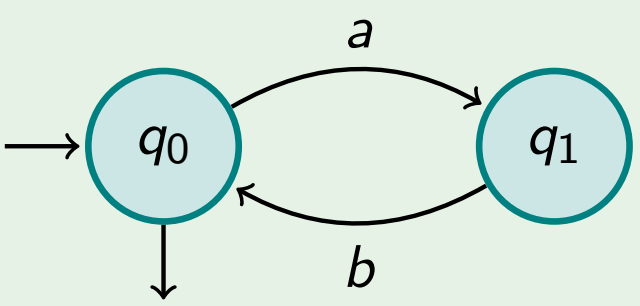

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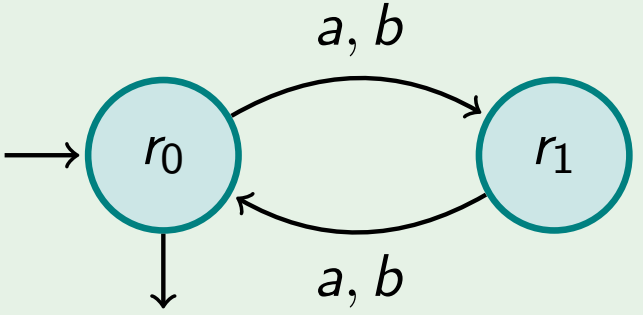

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This language is **not** star-free

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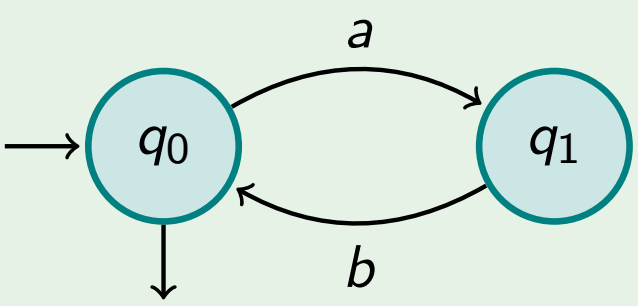

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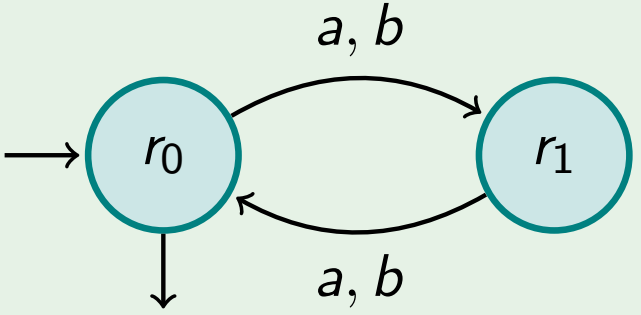

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Being star-free is a **semantic property** of a language.
Not obvious from the syntax that defines it.

Schützenberger's Theorem (Schützenberger'65)

Given a regular language L, the following are equivalent:

1. L is **star-free**
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The counter-free condition

McNaughton-Papert-Schützenberger

Given L regular, the following are equivalent:

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Counters

A **counter** is a **non-trivial** sequence of states $q_0, \ldots, q_n$ $(n \geq 1)$ along with a word w such that,

$$q_0 \xrightarrow{w} q_1 \xrightarrow{w} q_2 \xrightarrow{w} \cdots \xrightarrow{w} q_n \xrightarrow{w} q_0$$

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Easy to bound the length of the sequences and words we have to look for, which **yields decidability**.

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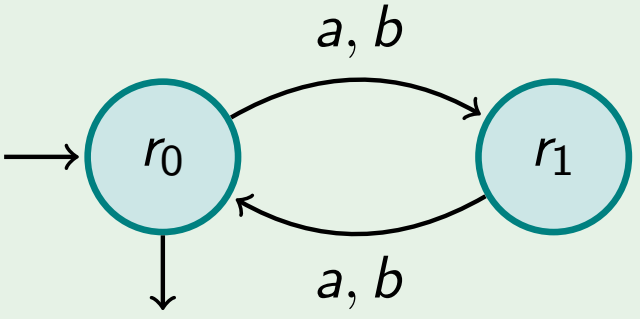

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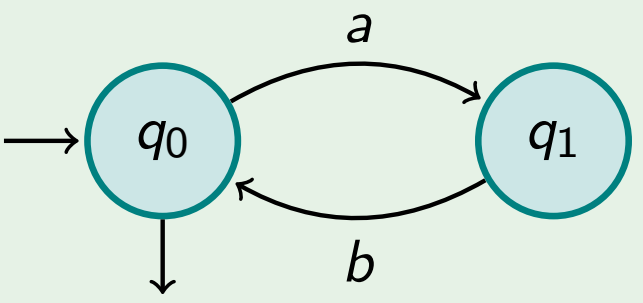

Language $(ab)^*$

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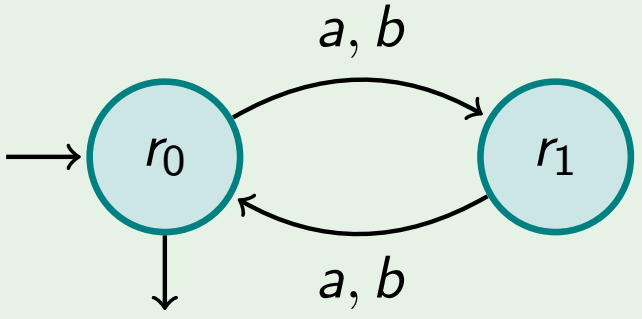

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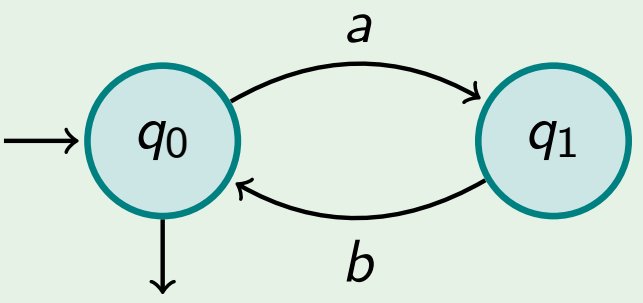

Language $(ab)^*$

Has a counter: not star-free

$$r_0 \xrightarrow{a} r_1 \xrightarrow{a} r_0$$

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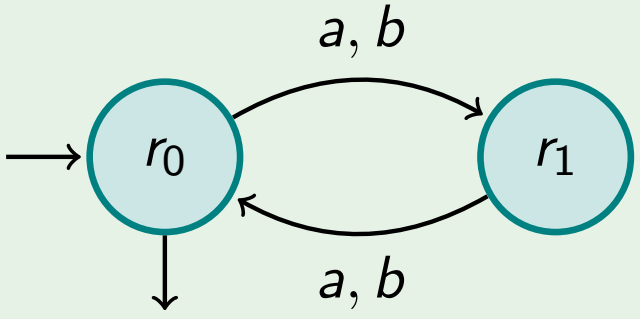

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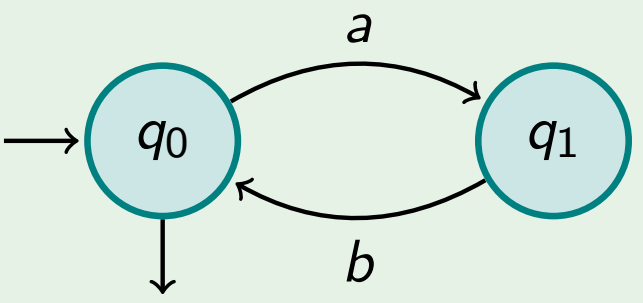
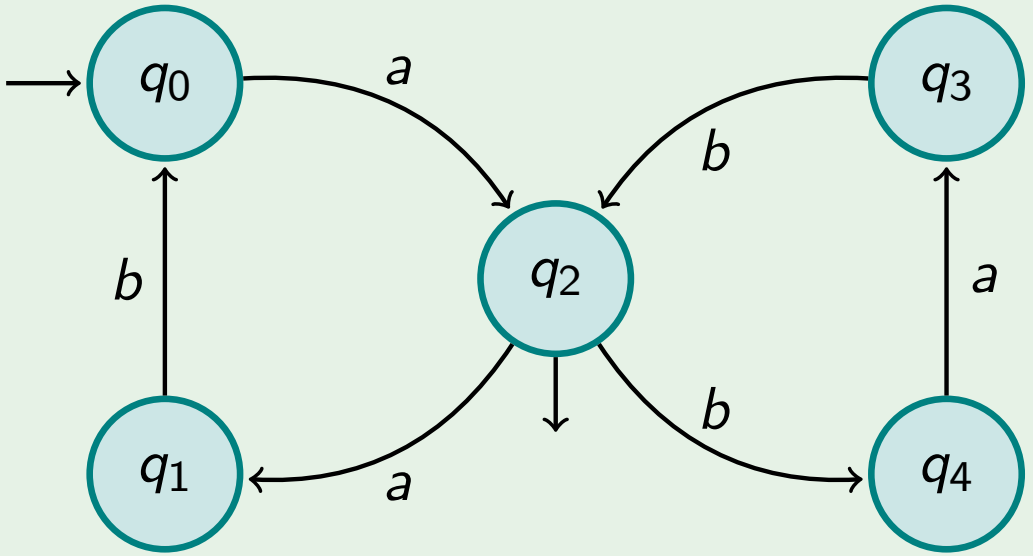


Language $(ab)^*$

Has no counter: star-free

Examples (2)



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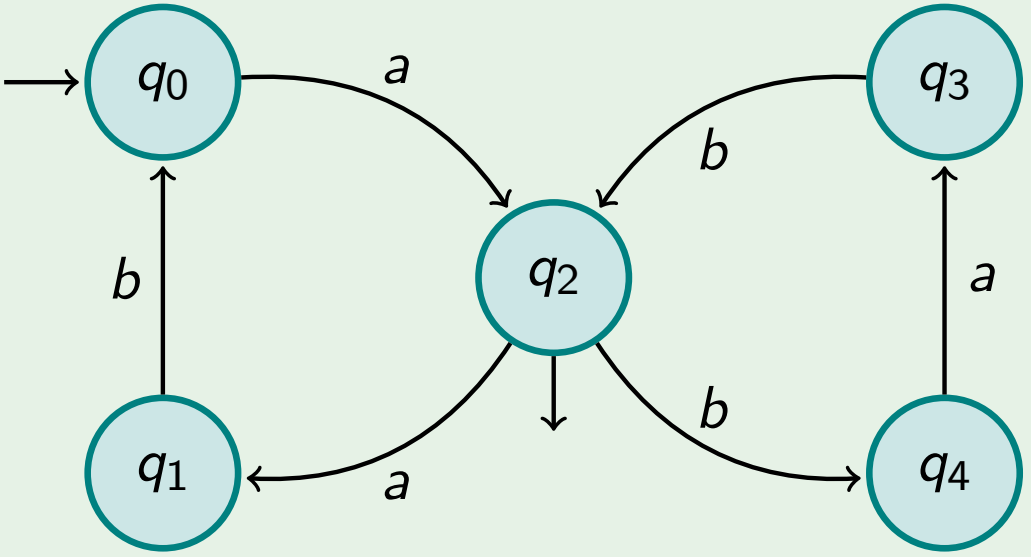

Has a counter: not star-free

$$q_0 \xrightarrow{ab} q_4 \xrightarrow{ab} q_2 \xrightarrow{ab} q_0$$

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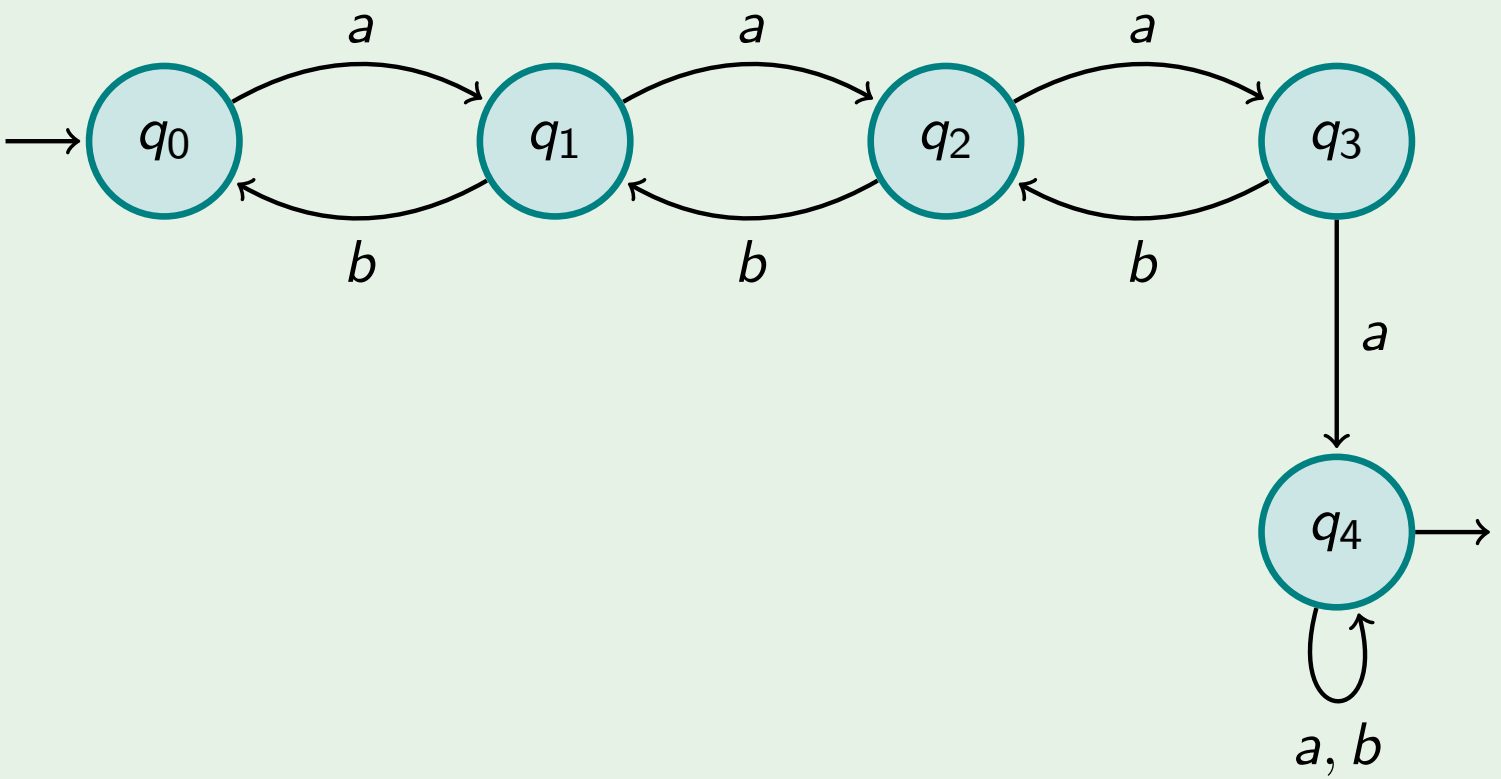

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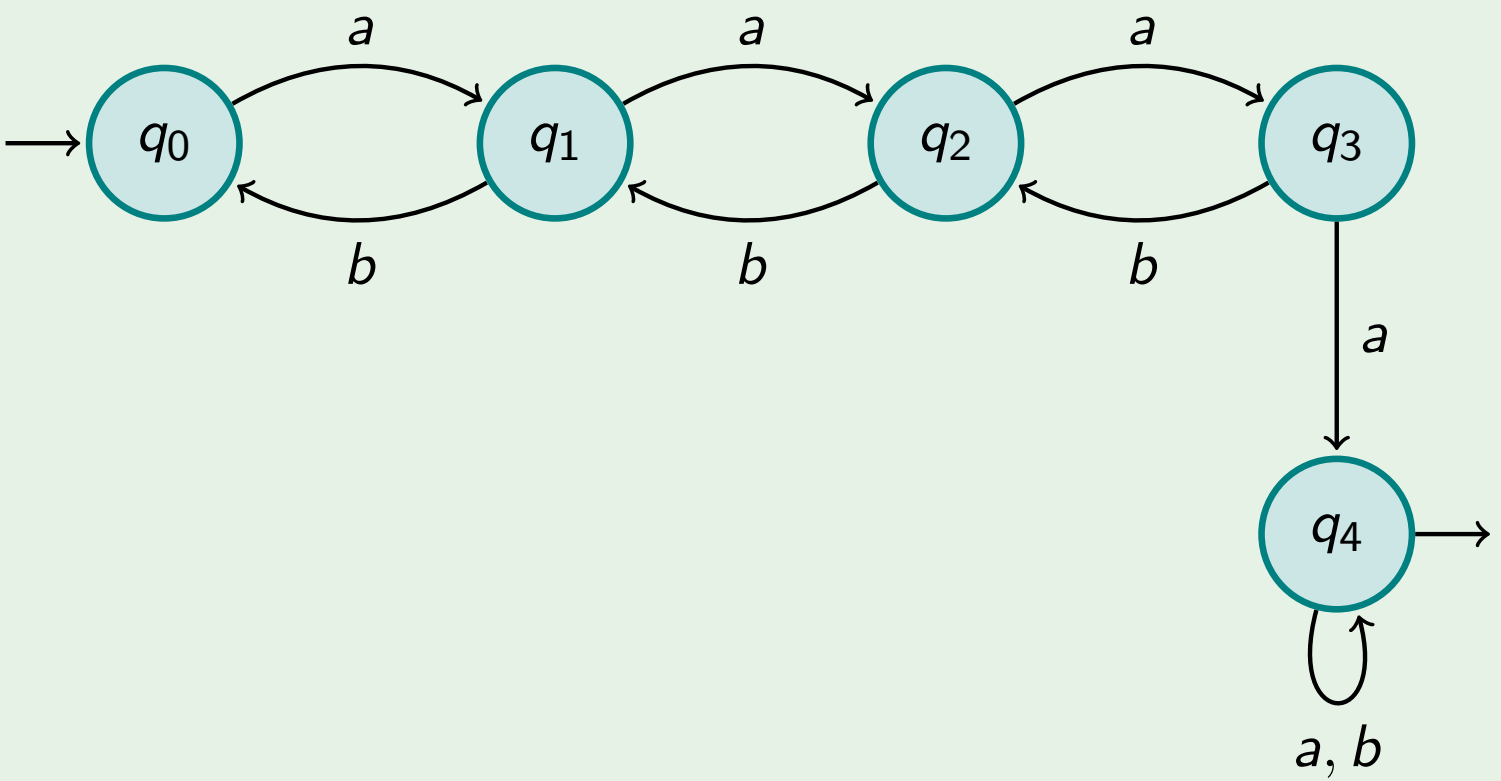

No counter: star-free

Good luck finding a star-free description

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  $\Rightarrow$ We get **normal forms** for star-free descriptions

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Why is it interesting ?

- A **Effective** characterization of the star-free languages.
- B **Generic construction** of star-free descriptions.

$\Rightarrow$ Altogether, we **learn a lot** about the star-free languages.

Why should we care about star-free languages ?

Complements to Schützenberger's Theorem

The class of star-free languages is important: it admits other definitions.

Given a regular language L, the following properties are equivalent:

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semantic

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The benefits of the theorem apply to both FO and LTL .

More about First-Order Logic (FO)

$$a\ b\ b\ b\ c\ a\ a\ a\ c\ a \in A^*$$
$$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9$$

- A word is a sequence of labeled positions that can be quantified.

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- One can test the linear order: $x < y$.

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- One can test the linear order: $x < y$.

$$\forall x\ (a(x) \Rightarrow \exists y\ (b(y) \wedge (y < x)))$$
"for any a in the word, there is a b to its left"

Each sentence defines a language
$\Rightarrow$ FO defines a class of languages.

This class is the class of star-free languages.

Defining Languages with First-Order Logic (FO)

Why are all star-free languages FO definable ?

- Finite and co-finite languages are FO-definable.
- Union, Intersection and Complement correspond to boolean connectives: $\vee, \wedge, \neg$.

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$w \in L_1 \cdot L_2$

There exists a position x in w such that:

- the prefix made up of all positions $y \leq x$ belongs to L_1.
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Proving the converse inclusion is more technical.

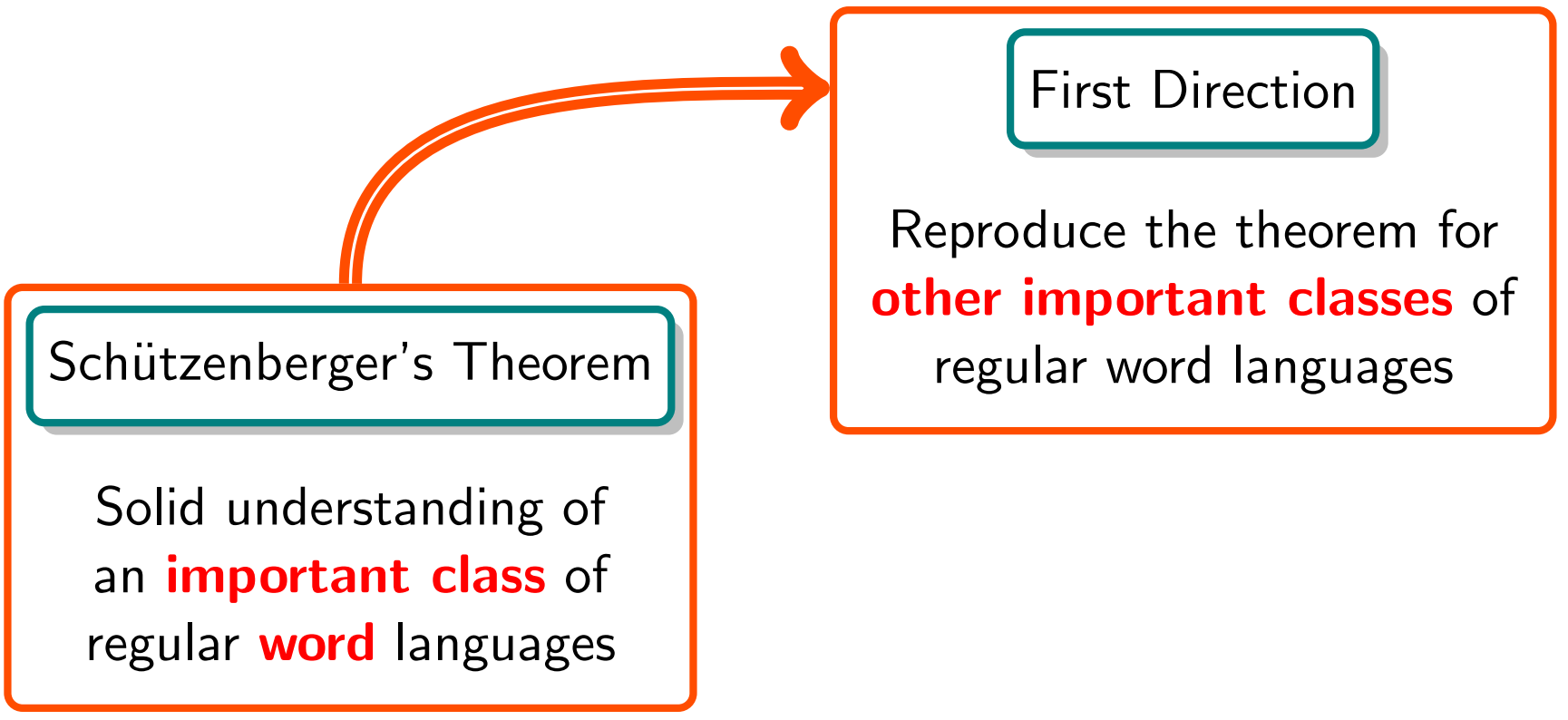
What is Next ?

After Schützenberger's Theorem

Schützenberger's Theorem

Solid understanding of
an **important class** of
regular **word** languages

After Schützenberger's Theorem



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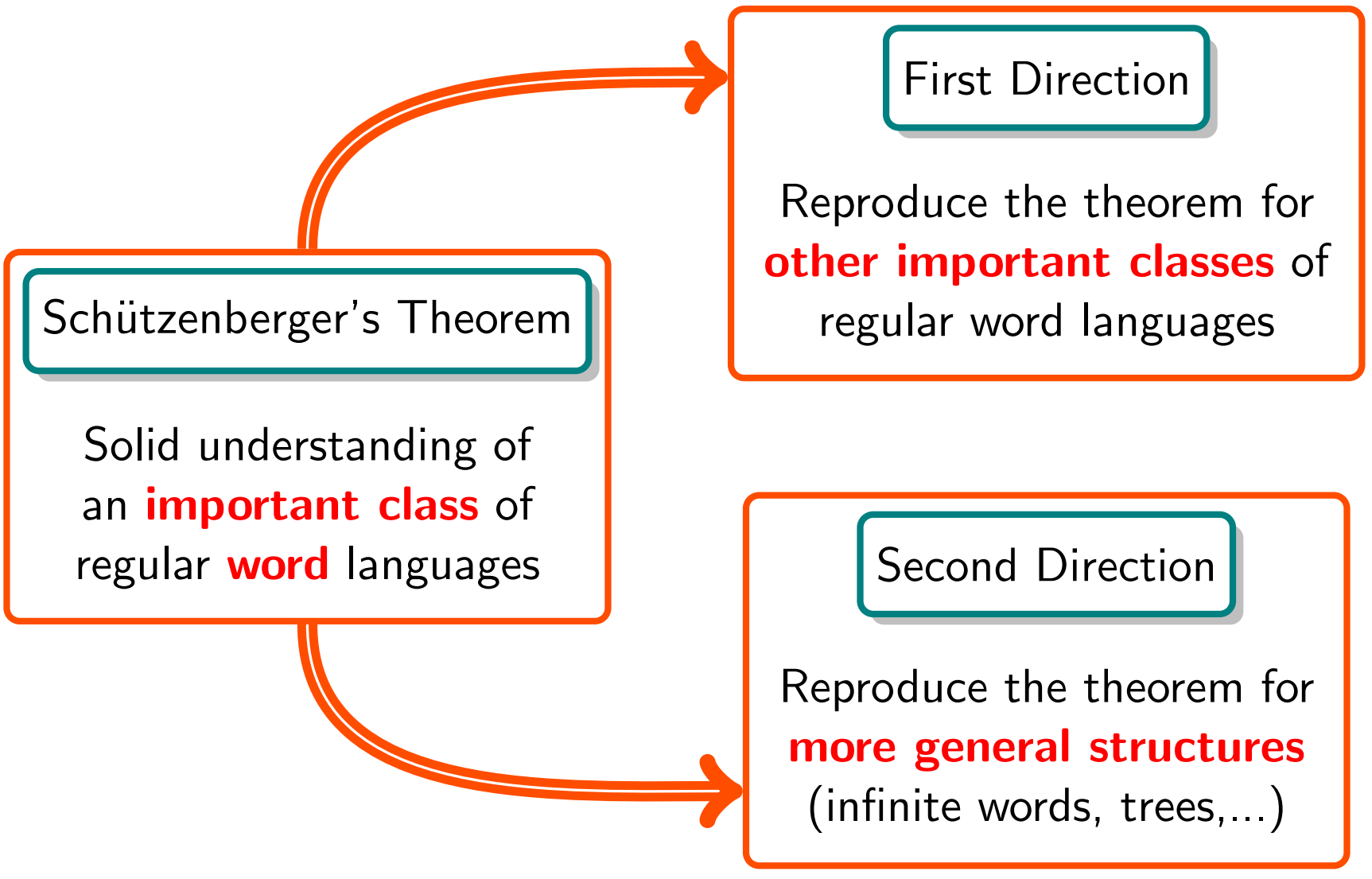

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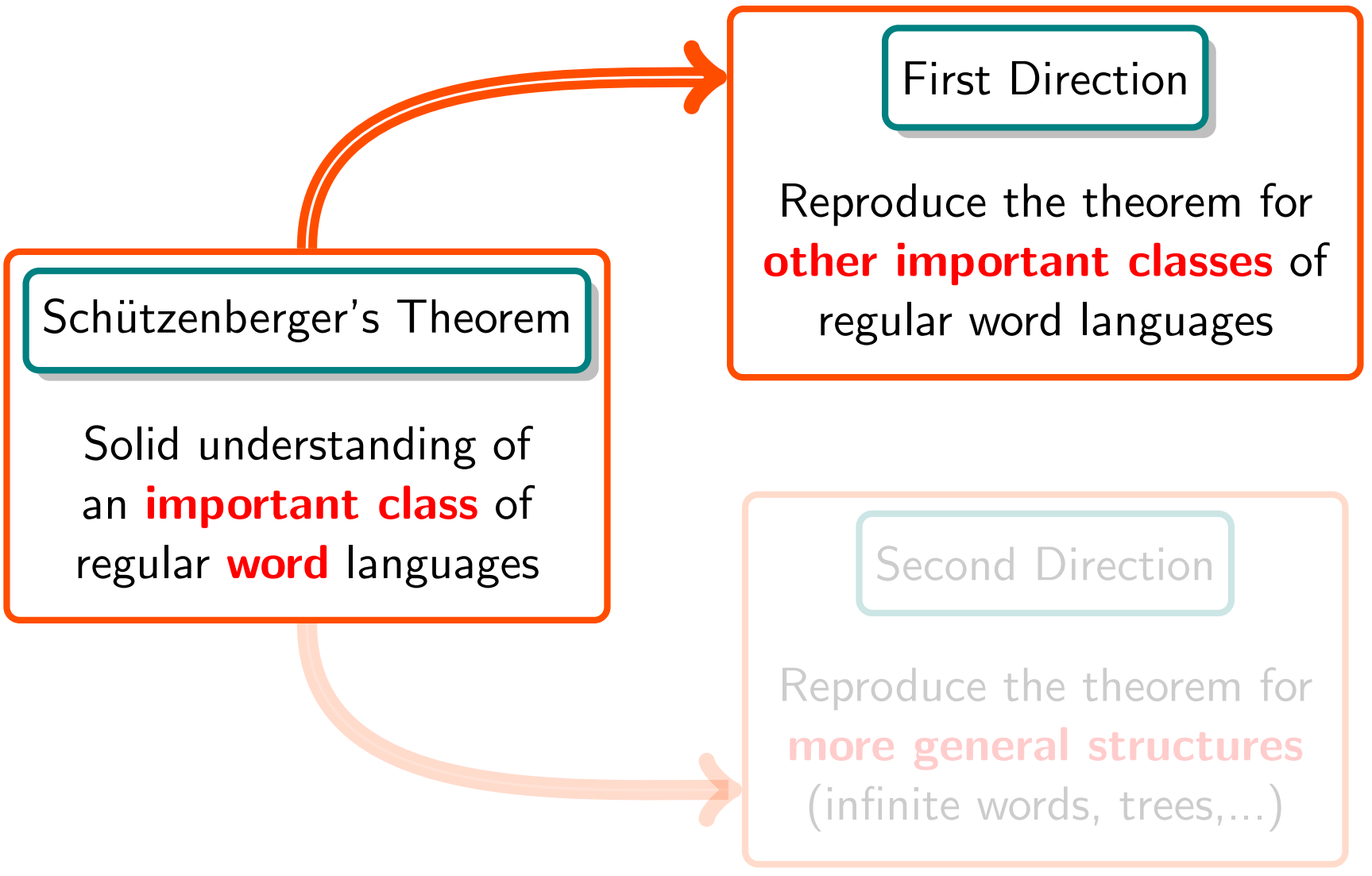
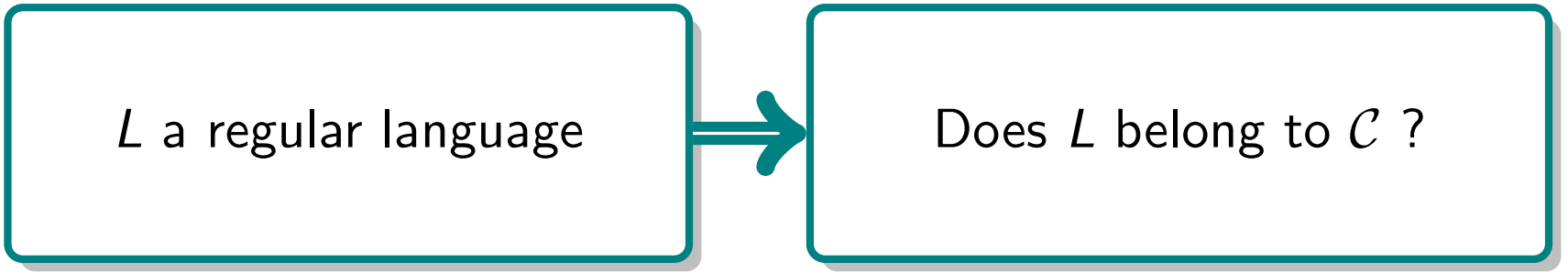


General Objective: Membership Problem

Reproducing Schützenberger's Theorem for other classes of languages amounts to considering the following problem.

Given such a class $\mathcal{C}$, the goal is to solve the associated **membership problem**:



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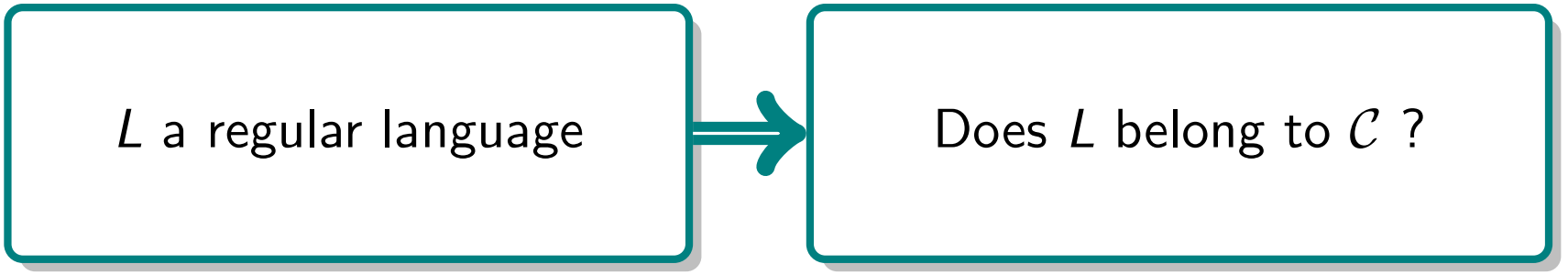

Given a particular class, there are two incremental goals to the problem:

- Objective 1: get an **algorithm** that decides it.
- Objective 2: find a generic way to **construct a sentence** that defines L when it exists.

Concatenation Hierarchies and Quantifier Alternation

What makes a star-free description complicated ?

Simple Language

$A^* \cdot aba \cdot A^*$

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$A^* \cdot a \cdot (\overline{A^* \cdot aba \cdot A^*} \cap \overline{A^* \cdot bab \cdot A^*}) \cdot a \cdot A^*$

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Complicated = alternation **concatenation** – **complementation**

Formalized by concatenation hierarchies.

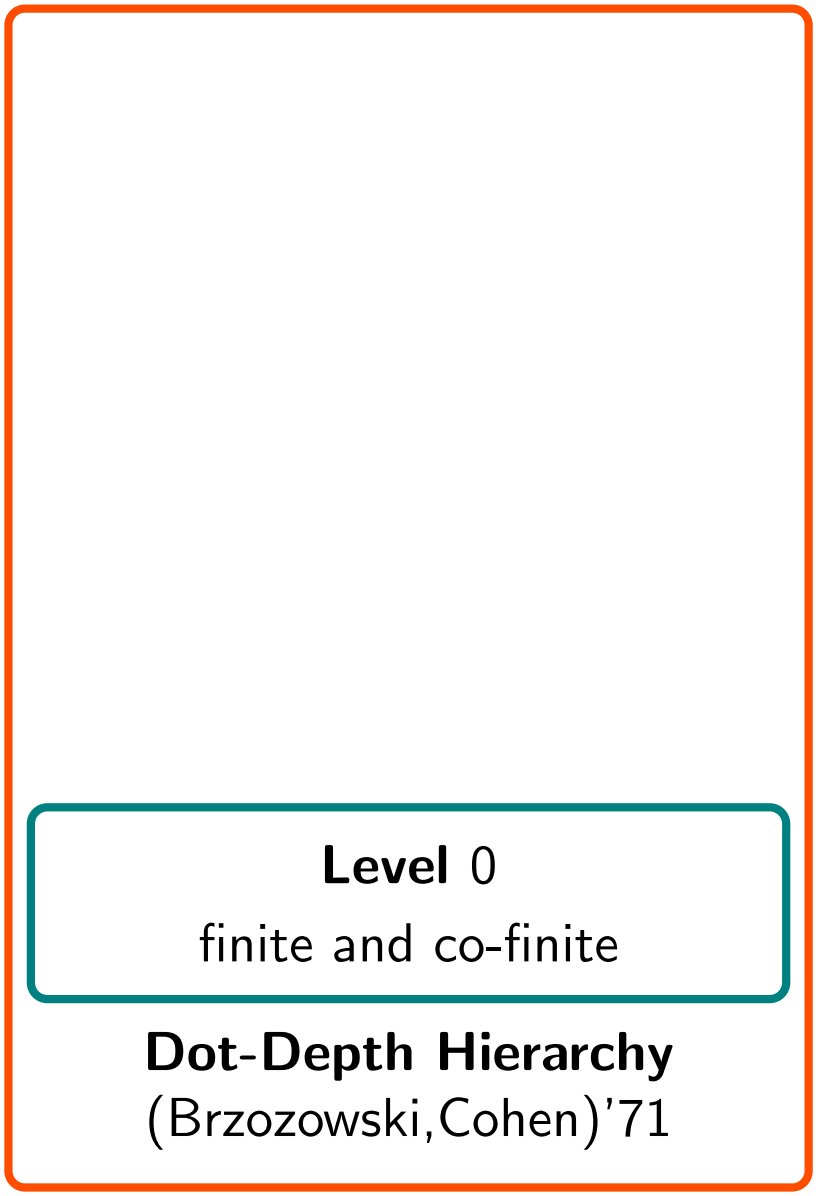
Classifying Star-free Languages (1)

The idea was formalized by two (closely related) infinite hierarchies which classify star-free languages into levels.

$$0 \text{ — } \tfrac{1}{2} \text{ — } 1 \text{ — } \tfrac{3}{2} \text{ — } 2 \text{ — } \tfrac{5}{2} \text{ — } 3 \text{ — } \tfrac{7}{2} \text{ — } 4 \cdots\cdots\cdots\cdots$$

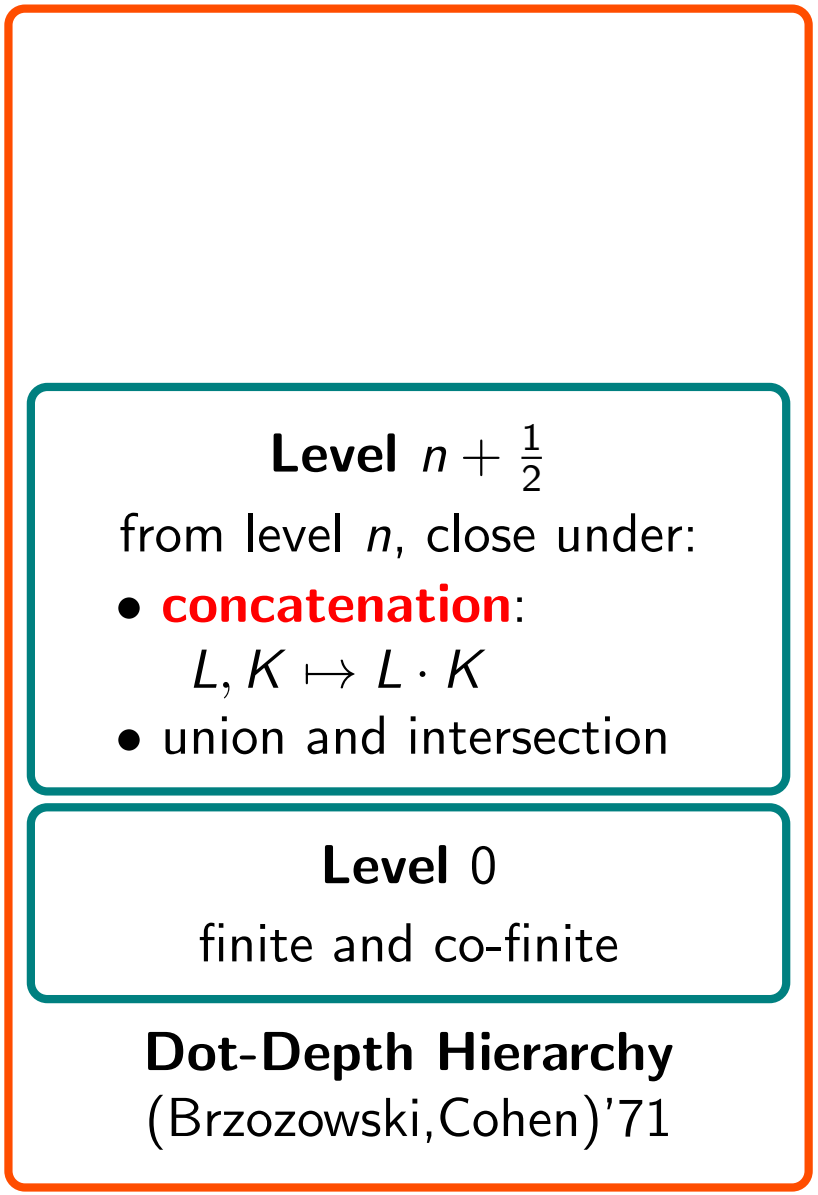
1. one goes from level i to level $i + \frac{1}{2}$ using concatenation.
2. one goes from level $i + \frac{1}{2}$ to level $i + 1$ using complementation.

Classifying Star-free Languages (2)



Examples:
A^*, $\{aa\}$, $\{b\}$, $\{aa, bb\}$,...

Classifying Star-free Languages (2)



Examples for level $\frac{1}{2}$:
$A^* \cdot aa \cdot A^*, A^* \cdot ab \cdot A^* \cup b \cdot A^* \cdot a, \ldots$

Examples:
A^*, $\{aa\}$, $\{b\}$, $\{aa, bb\}, \ldots$

Classifying Star-free Languages (2)

Level $n+1$

from level $n+\frac{1}{2}$, close under:

- **complementation**,
- union and intersection

Level $n+\frac{1}{2}$

from level n, close under:

- **concatenation**:
  $L, K \mapsto L \cdot K$
- union and intersection

Level 0

finite and co-finite

Dot-Depth Hierarchy
(Brzozowski,Cohen)'71

Example for level 1:

$$\overline{b \cdot A^*} \cap \overline{A^* \cdot aa \cdot A^*} \cap \overline{A^* \cdot bb \cdot A^*} \cap \overline{A^* \cdot a}$$

Examples for level $\frac{1}{2}$:

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Level $n+\frac{1}{2}$

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  $L, K, a \mapsto L \cdot a \cdot K$
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Straubing-Thérien Hierarchy
(Straubing)'81 (Thérien)'81

Both hierarchies are know to be **strict**,
(Brzozowski,Knast)'78

A natural objective

Objective

Precisely understand what can be defined with "simple" star-free descriptions:

Solve **membership for each level** in both hierarchies.

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Before talking about this objective, let us motivate it further:

The hierarchies also admit a **natural logical definition**.

Connection with Logic: Quantifier Alternation

What is a simple FO sentence ?

$$\forall x \ (a(x) \Rightarrow \exists y \ (b(y) \wedge (y < x)))$$

What is a complicated sentence ?

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$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \forall x_6 \forall x_7 \exists x_8 \ \varphi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$

Complicated = High Quantifier Alternation

Connection with Logic: Quantifier Alternation

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$$\forall x \ (a(x) \Rightarrow \exists y \ (b(y) \wedge (y < x)))$$

$\Rightarrow \Pi_2$ sentence.

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$\Rightarrow \Sigma_7$ sentence (φ quantifier-free)

Complicated = High Quantifier Alternation

FO Quantifier Alternation Hierarchy

Level n: Σ_n

For all n, a Σ_n sentence is (in prenex normal form)

$$\underbrace{\exists x_1, \ldots, x_{n_1} \forall y_1, \ldots, y_{n_2} \cdots\cdots}_{n \text{ blocks (starting with } \exists)} \quad \underbrace{\varphi(\bar{x}, \bar{y}, \ldots)}_{\text{quantifier-free}}$$

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Σ_n is not closed under complement $\Rightarrow$ we get two other classes:

Level n: Π_n

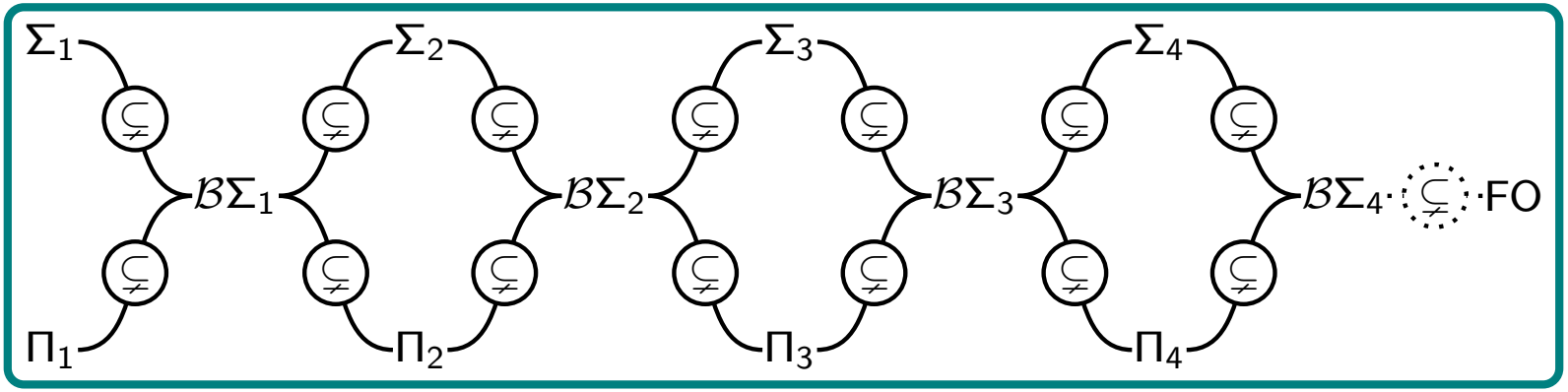
Negation of a Σ_n sentence:

$$\underbrace{\forall x_1, \ldots, x_{n_1} \exists y_1, \ldots, y_{n_2} \cdots}_{n \text{ blocks (starting with } \forall)} \quad \varphi$$

Level n: $\mathcal{B}\Sigma_n$

Boolean combinations of Σ_n (and Π_n) sentences.

FO Quantifier Alternation Hierarchy



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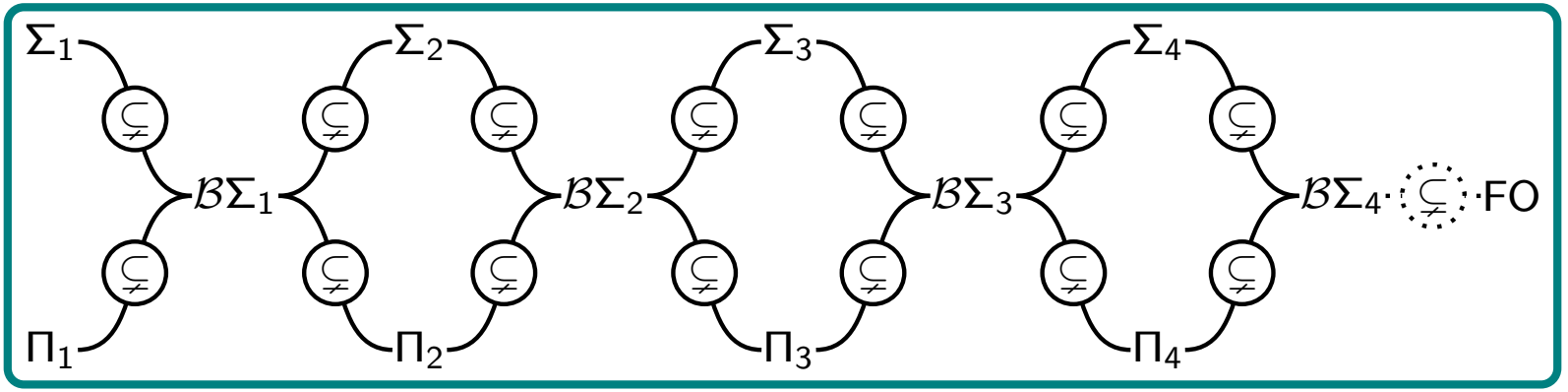

In the Straubing-Thérien Hierarchy (Perrin-Pin'86),

- level n is exactly $\mathcal{B}\Sigma_n$,
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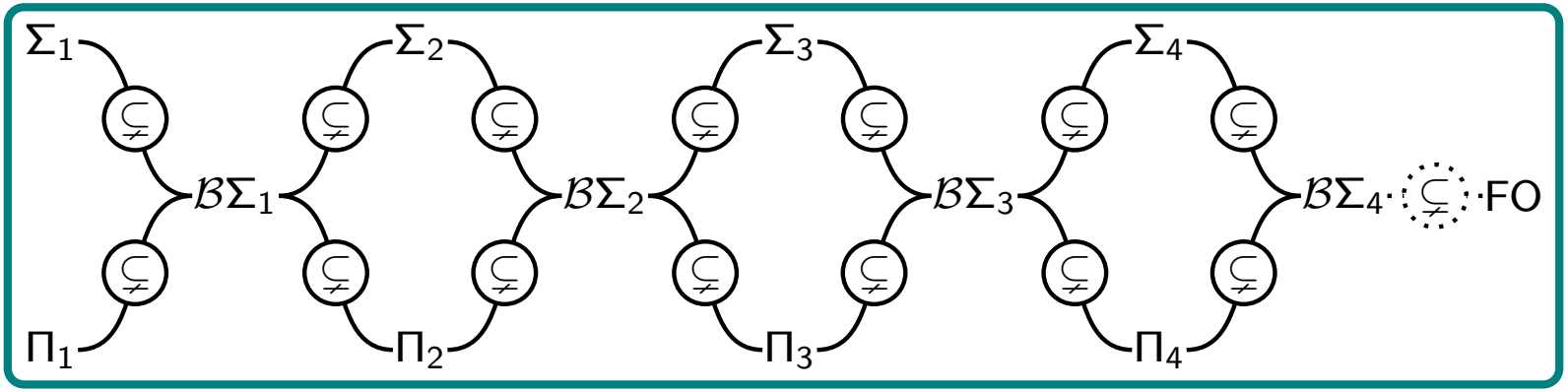

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Same correspondence between dot-depth and a variant of FO with an enriched signature (Thomas'82)

Membership and the Hierarchies

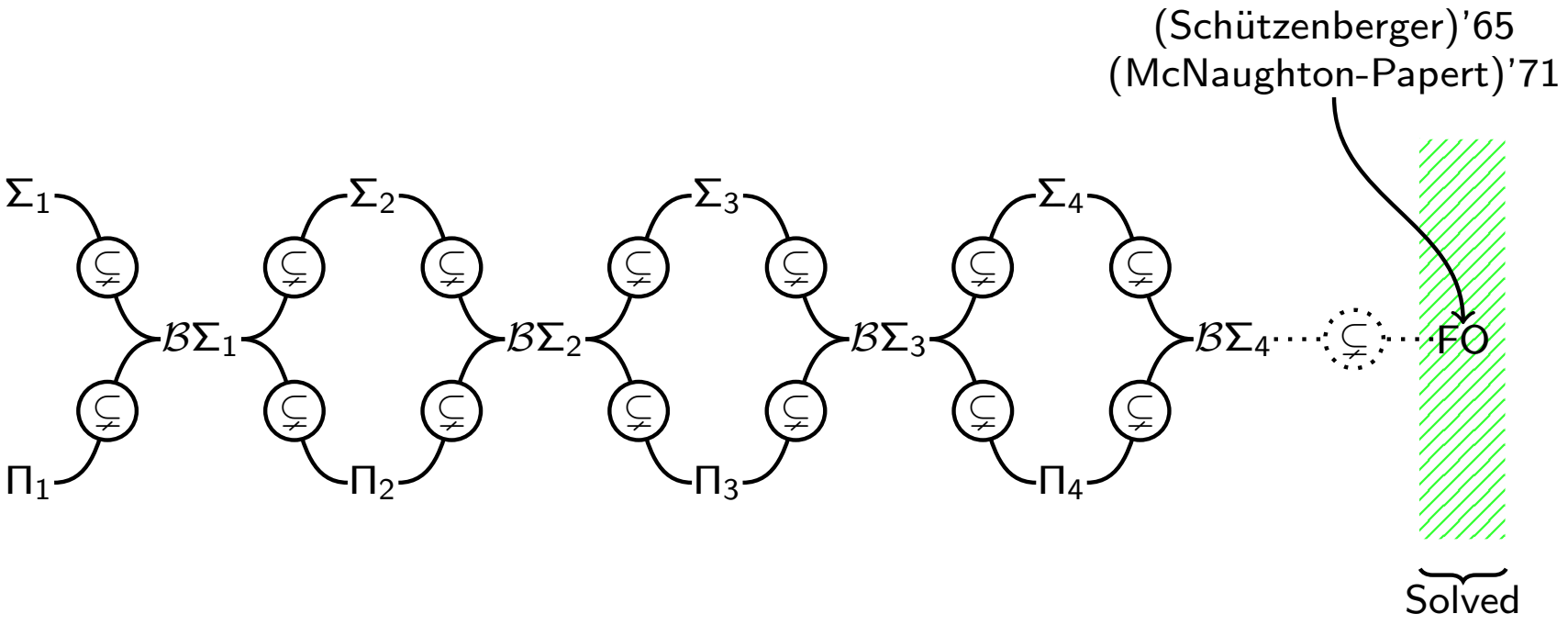
Remark

1. From now on, I will use the logical point of view (Σ_n, $\mathcal{B}\Sigma_n$).
2. Nowadays, people mostly focus on the Straubing-Thérien hierarchy. This is what I will do.

(Straubing'85) For each level:

membership for the dot-depth hierarchy
can be **effectively reduced**
to membership for the Straubing-Thérien Hierarchy

FO Quantifier Alternation: Membership State of the Art



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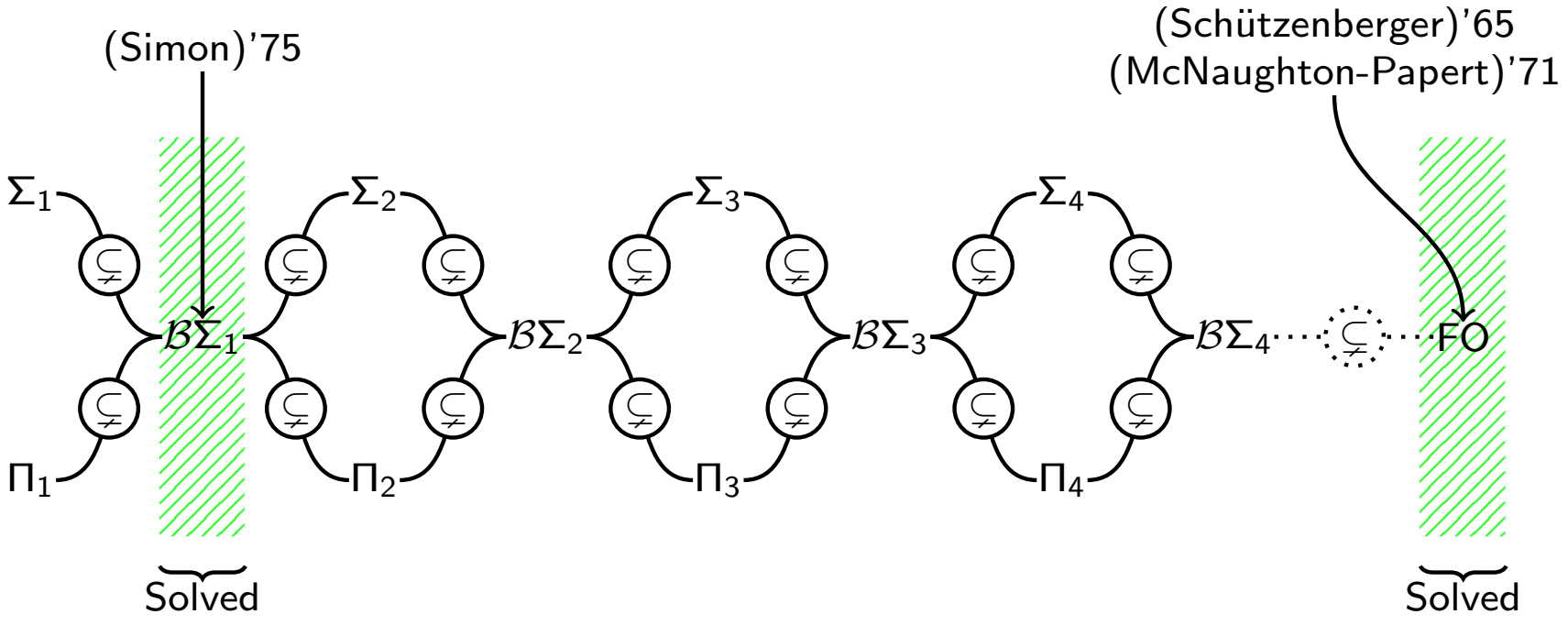

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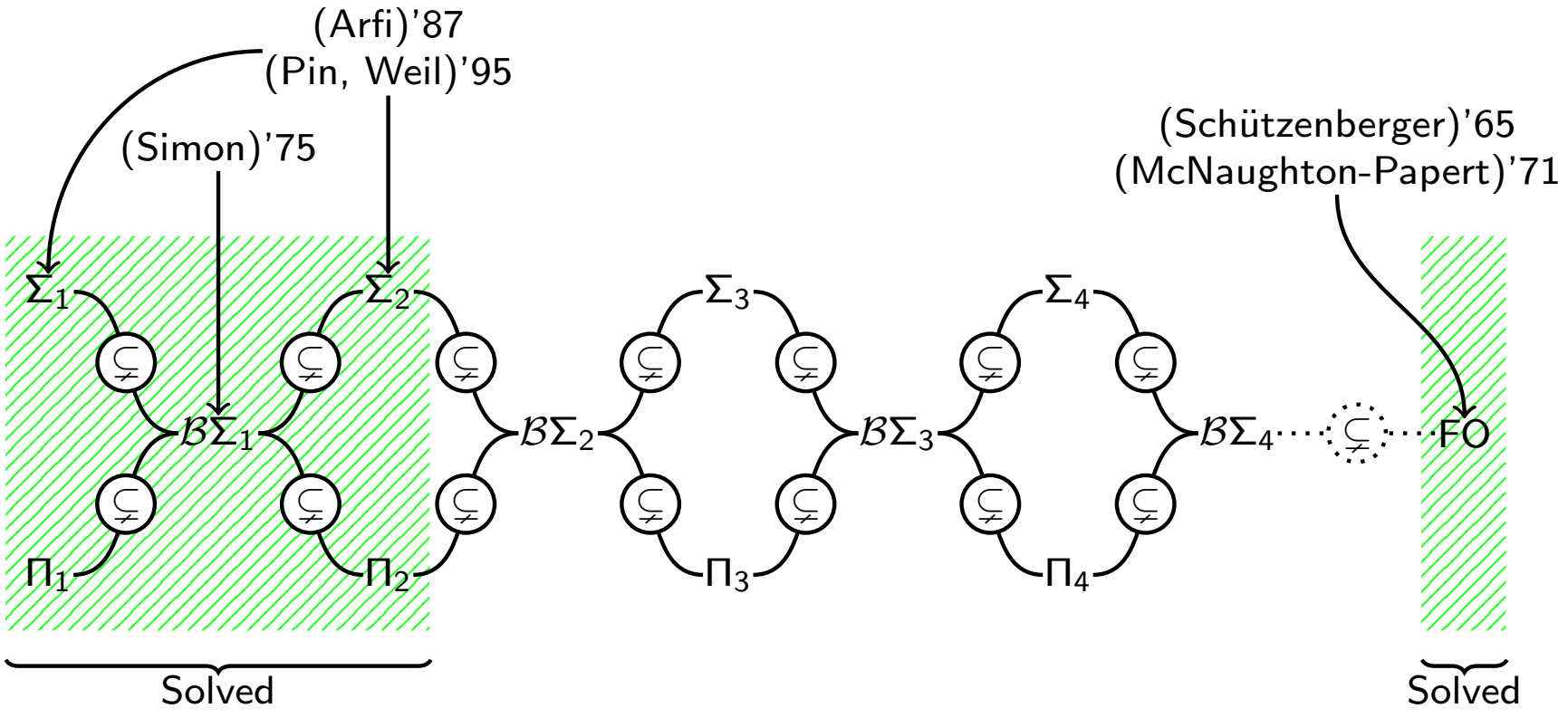

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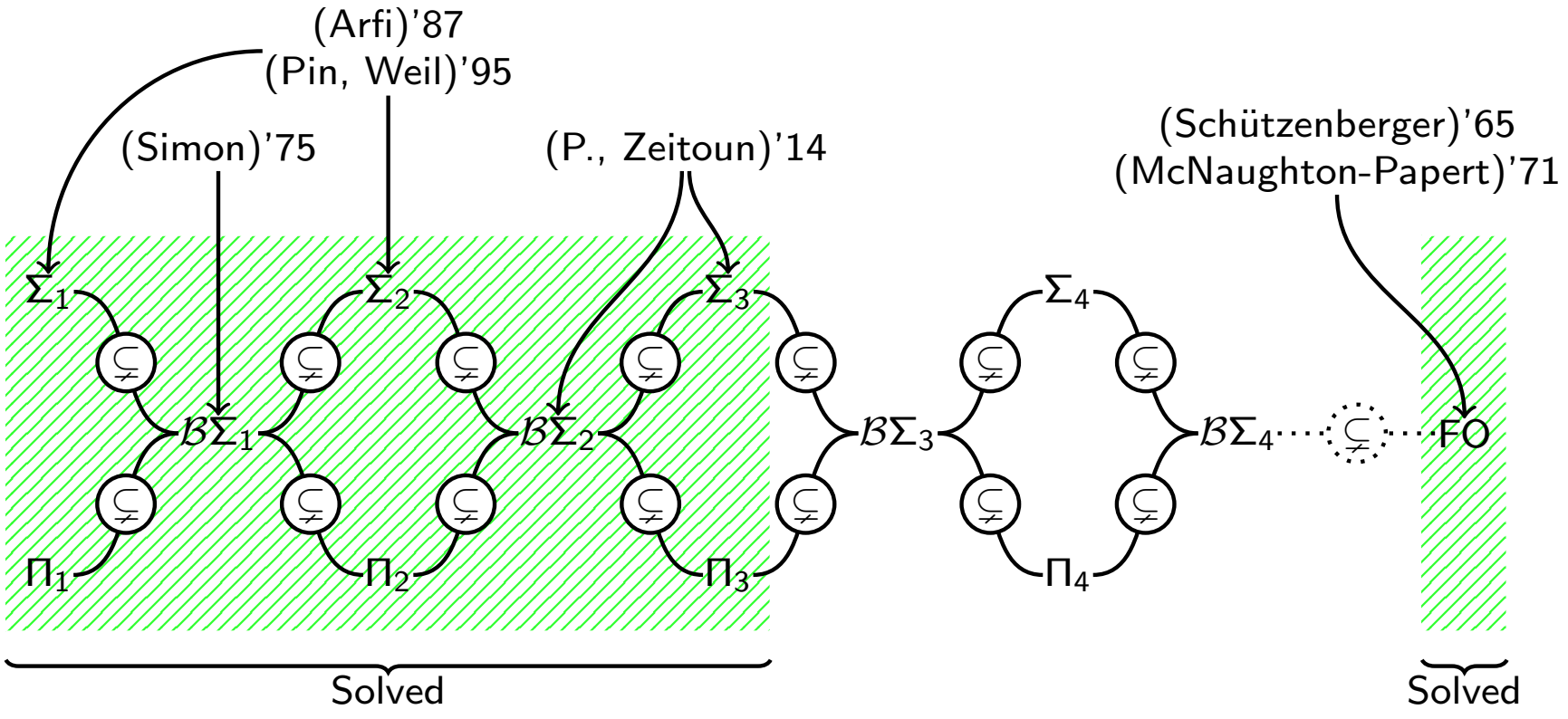

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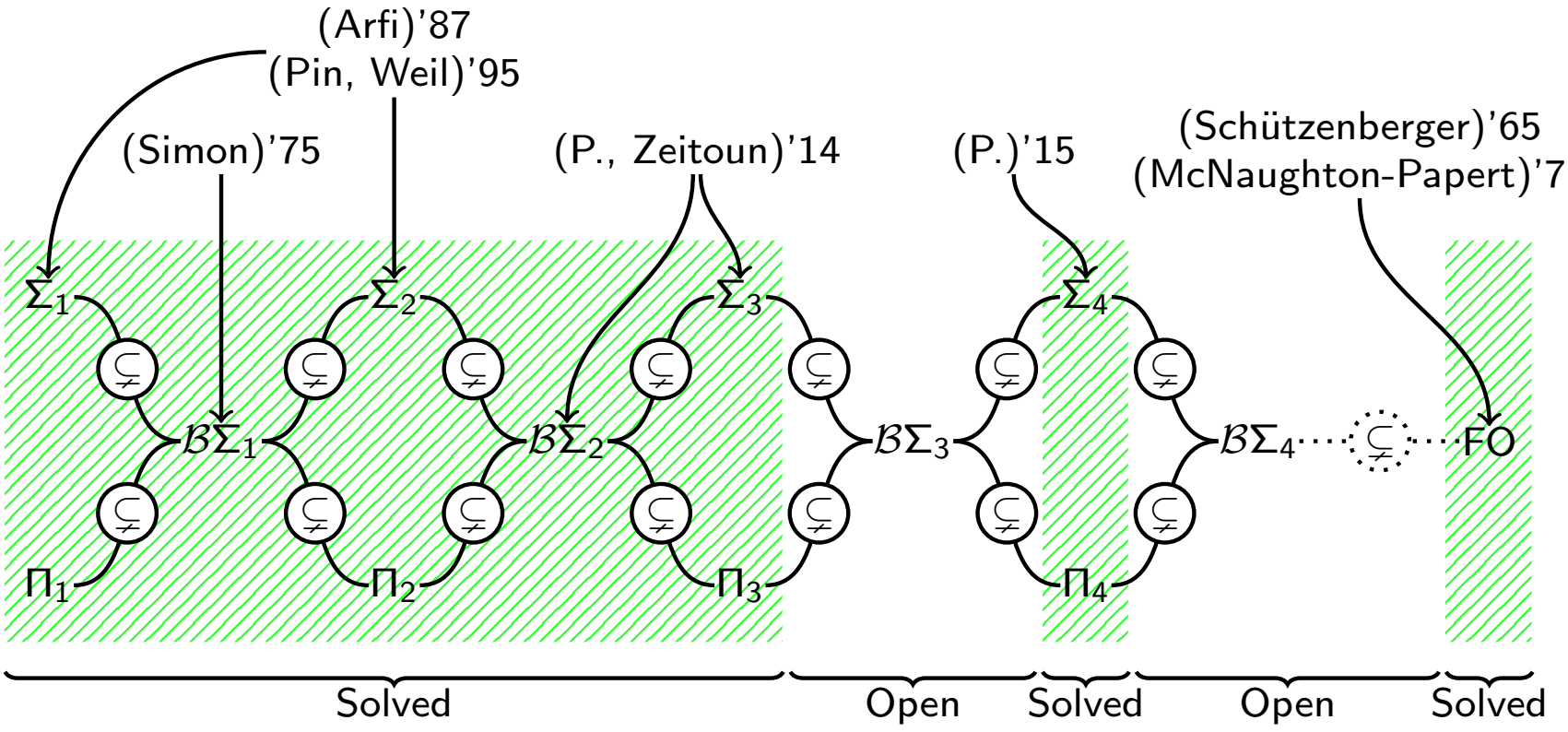

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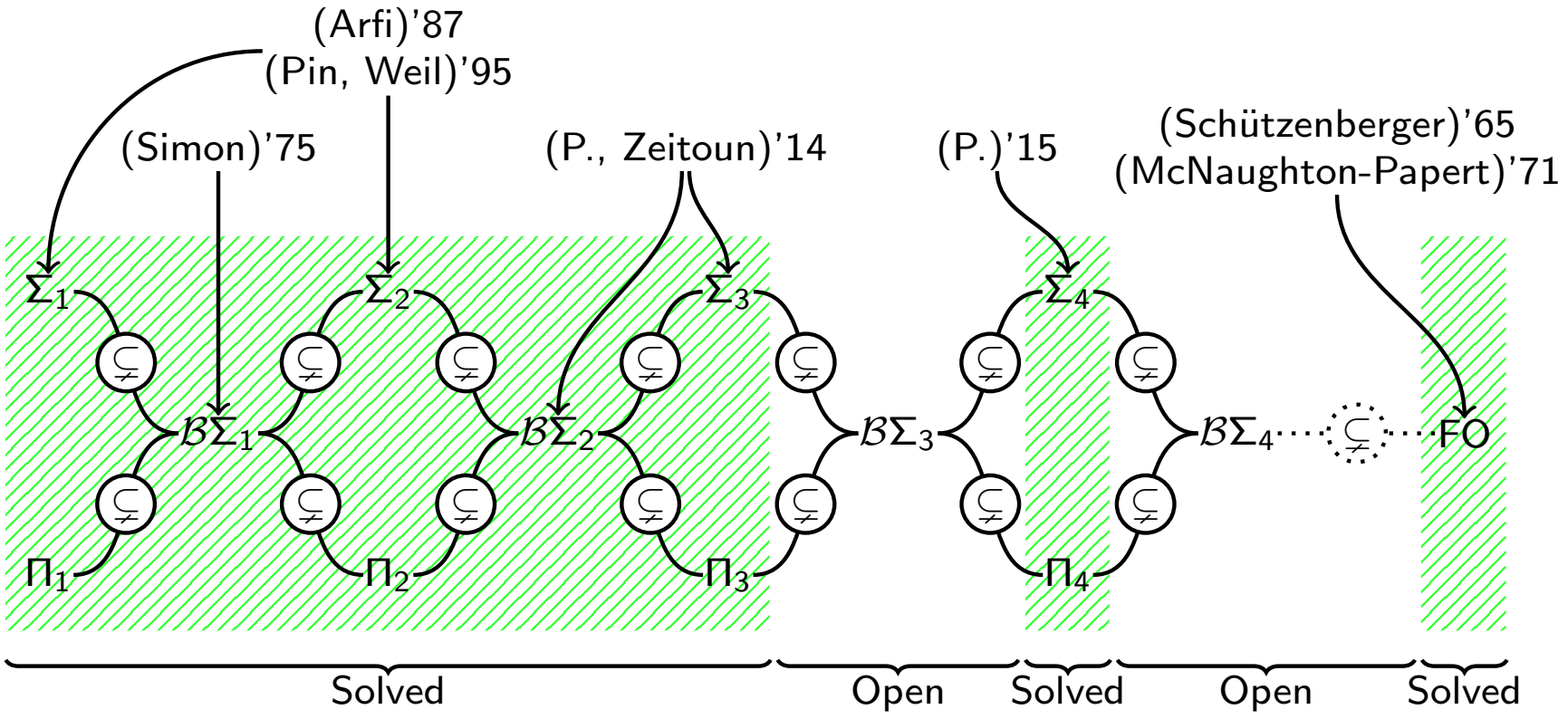


Why is progress on this question so slow ?

Why are we still stuck ?

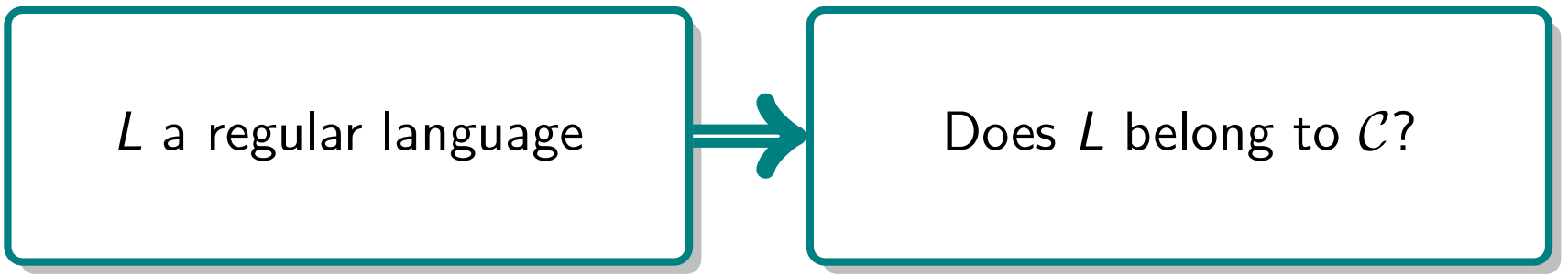
A deeper look at membership questions

Quick Remark

- The original approach uses an algebraic point of view: the central object is the syntactic monoid.
- I will use an automata point of view: the central object is the minimal automaton.

Membership Problem: General approach

For a level $\mathcal{C}$ in the hierarchy, we search for a membership algorithm for $\mathcal{C}$ which decides the following problem.



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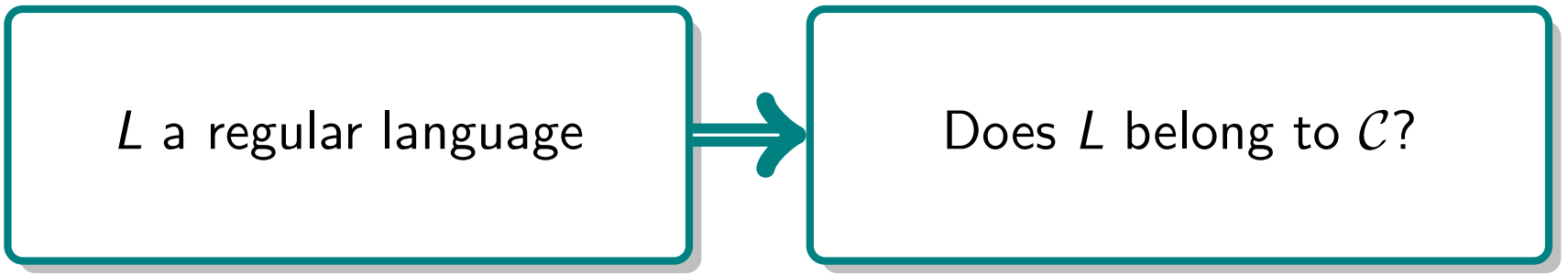
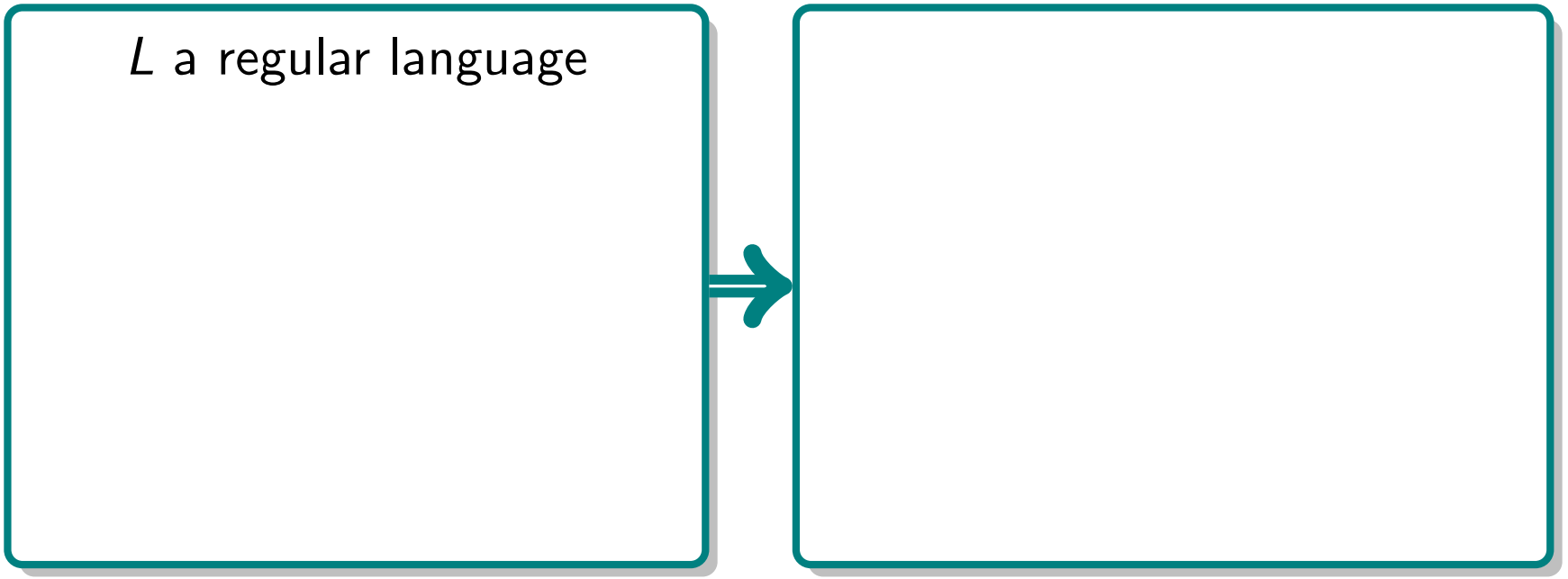


This is not how the question is approached.

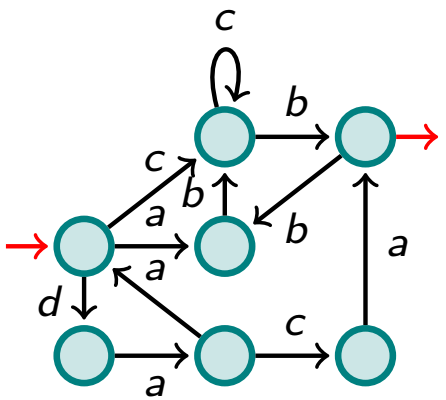
The problem we are really considering



The problem we are really considering

L a regular language

It has a **minimal automaton**



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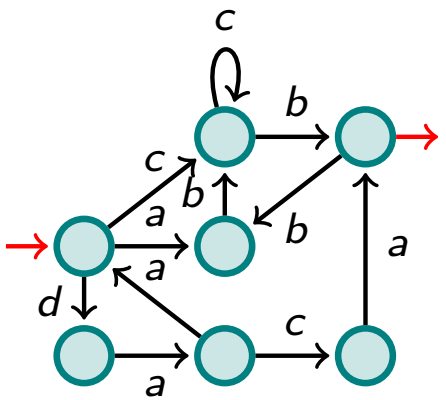

$\Rightarrow$ **set of accepted languages**

The initial and final states can be changed

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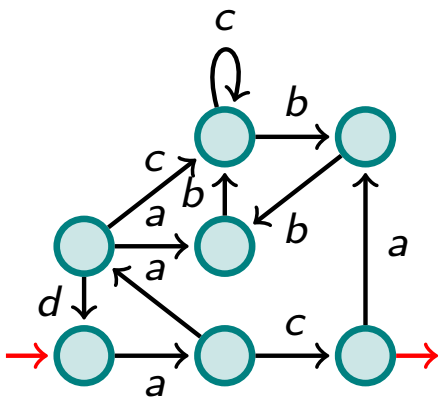

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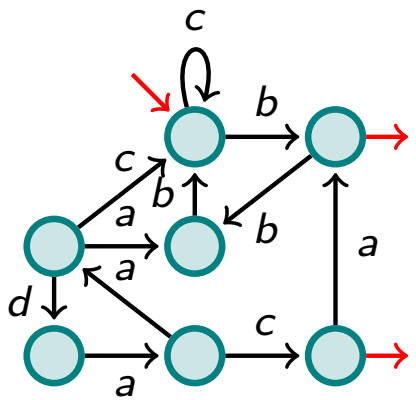

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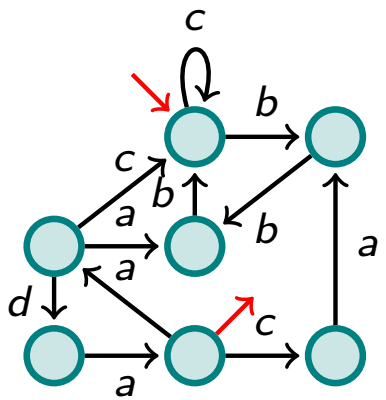

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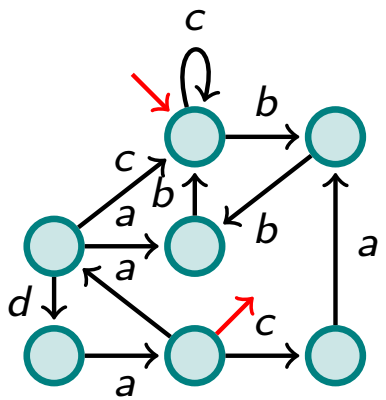

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L in $\mathcal{C}$

$\Updownarrow$

All accepted languages in $\mathcal{C}$

The problem we are really considering

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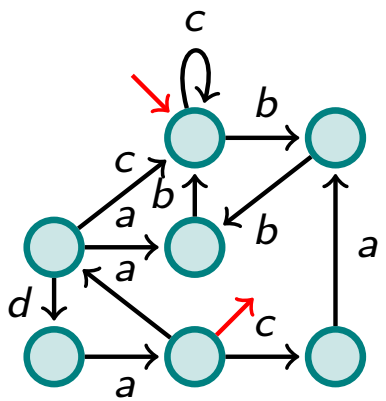


The initial and final states can be changed

$\Rightarrow$ **set of accepted languages**

L in $\mathcal{C}$

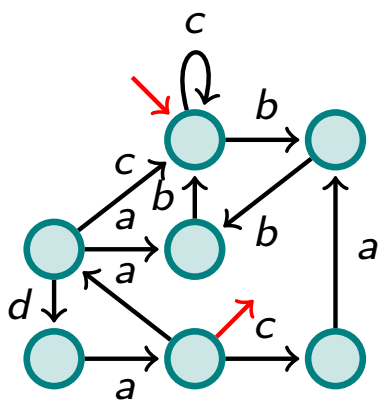
All accepted languages in $\mathcal{C}$

- The set is finite and has a **structure**.
- This structure is connected to our building operations: boolean operations and **concatenation**.
- L is built-up from the languages in the set.

The problem we are really considering

L a regular language

It has a **minimal automaton**



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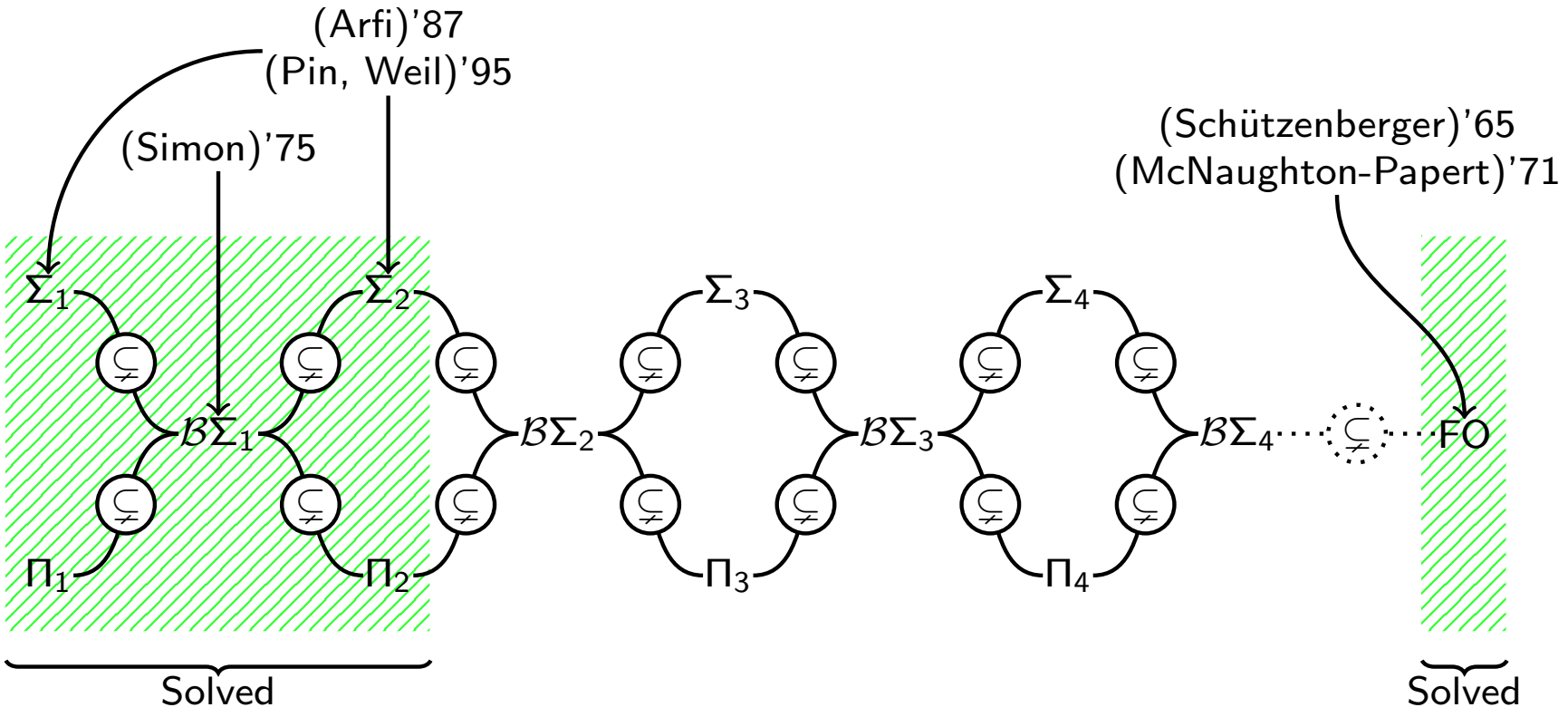
- This structure is connected to our building operations: boolean operations and **concatenation**.
- L is built-up from the languages in the set.

Algorithms (and their proofs) are **based on this structure**.

With this approach alone, we get the following results



What is the missing ingredient to go higher ?

Issues with the general approach

Reminder: the problem we are really considering

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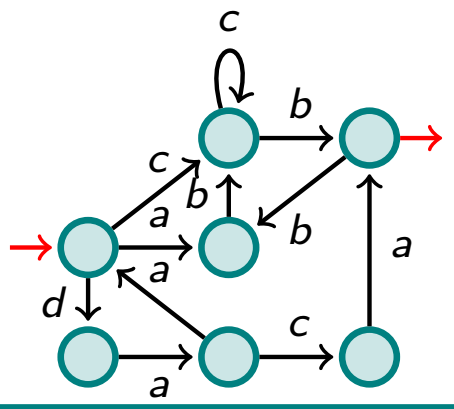

We ask whether **all** accepted languages belong to $\mathcal{C}$

L in $\mathcal{C}$

$\Updownarrow$

All accepted languages in $\mathcal{C}$

Benefits

Algorithms (and their proofs) are **based on the structure of the set of accepted languages**.

Reminder: the problem we are really considering

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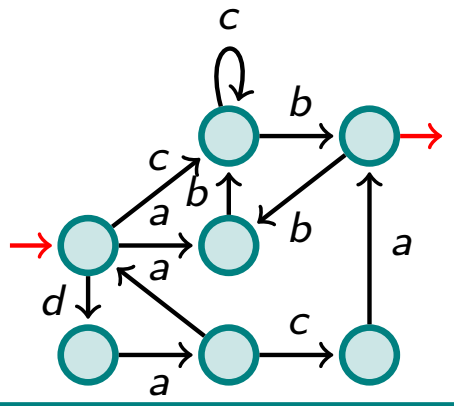


We ask whether **all** accepted languages belong to $\mathcal{C}$

L in $\mathcal{C}$ ⇕ All accepted languages in $\mathcal{C}$

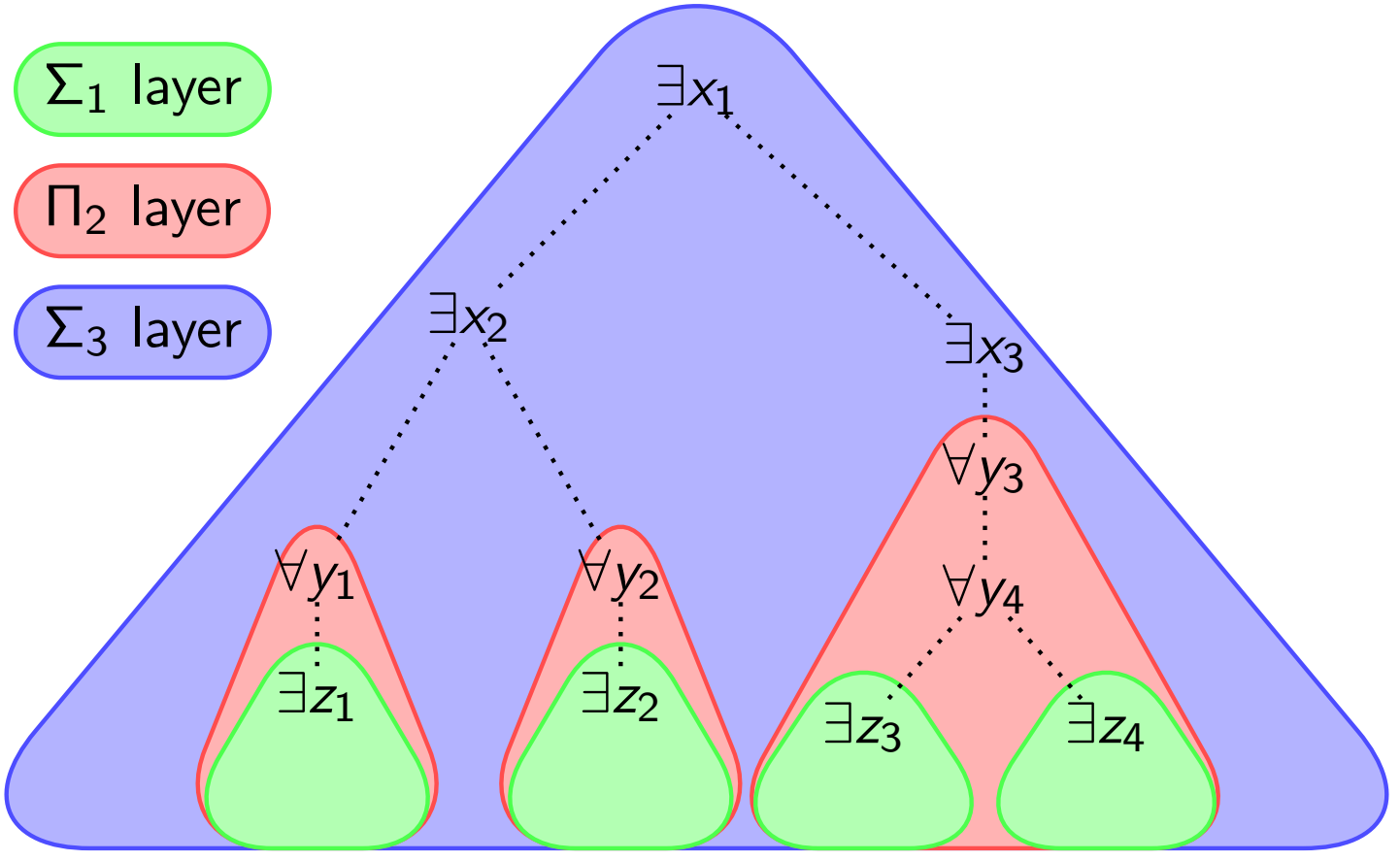
Benefits

Algorithms (and their proofs) are **based on the structure of the set of accepted languages**.

In particular, this is **essential for the inductive construction** of sentences.

Why is this a problem for the Hierarchy ?

A Σ_n sentence is **layered**: Consider a Σ_3 sentence



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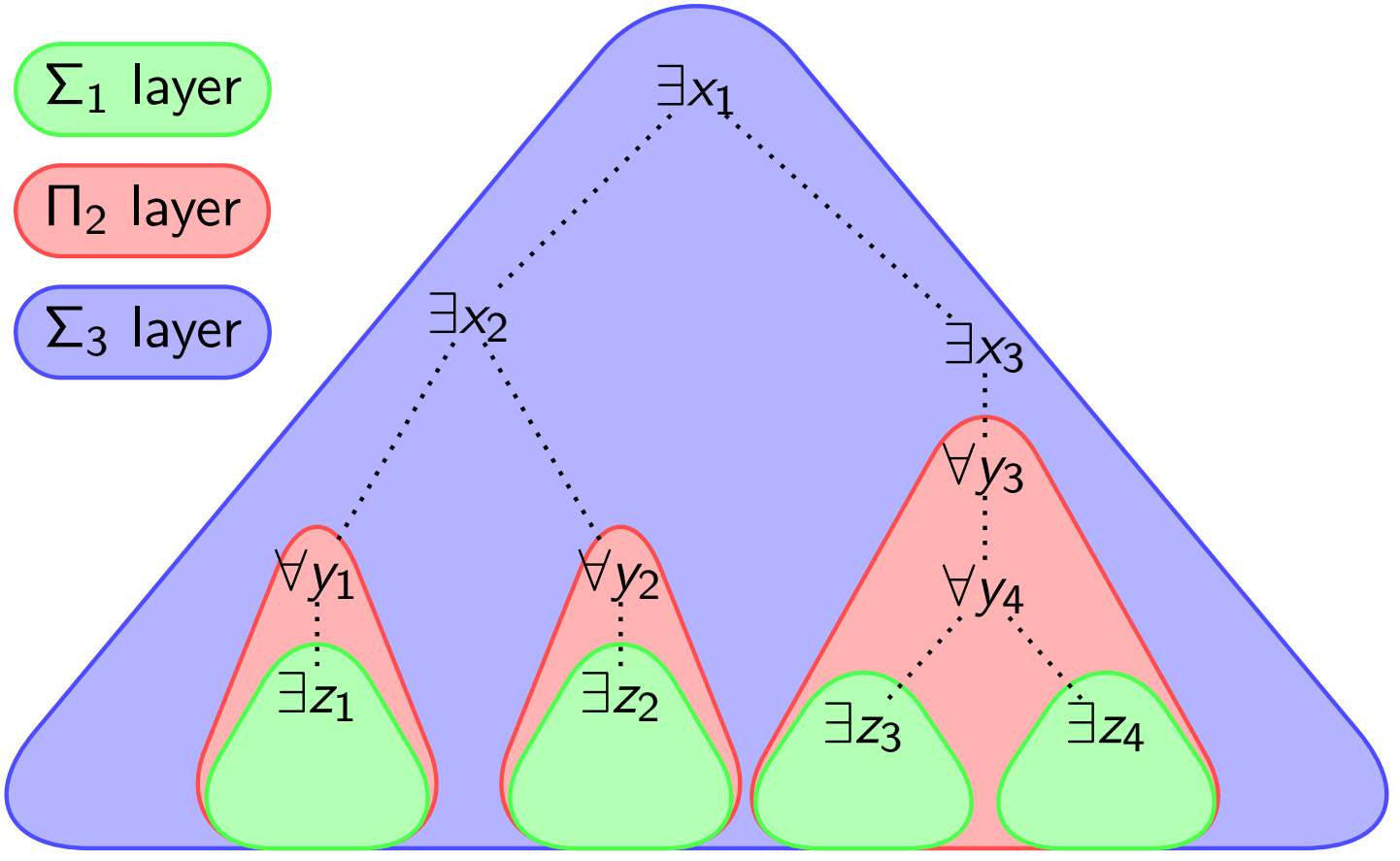

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$\Rightarrow$ We first need to ask a "Π_2 **question**" on our input automaton.

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- $\Rightarrow$ The "Π_2 question" must involve **a more general problem than membership**
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The main idea behind membership for $\mathcal{B}\Sigma_2$, Σ_3

Two steps:

1. solve a deeper problem for Σ_2 independently.
2. reuse the answer to this problem in the membership algorithm for $\mathcal{B}\Sigma_2$ or Σ_3.

What is a "more general problem than membership"?

What we want is an "**approximation problem**":

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We have several "approximation problems" (each one tailored to a particular logic). However they are all based on a common one:

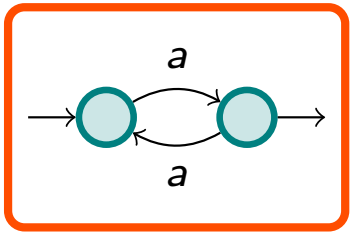
the **separation problem**

More General: Separation is Here

Given a level $\mathcal{C}$ in the hierarchy, decide the following problem:

L_1, L_2 two regular languages

L_1



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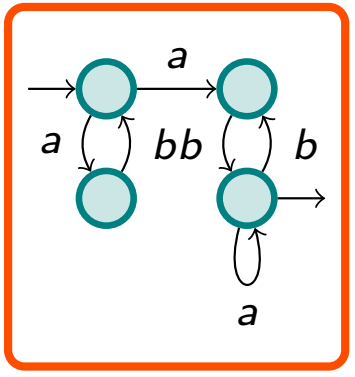

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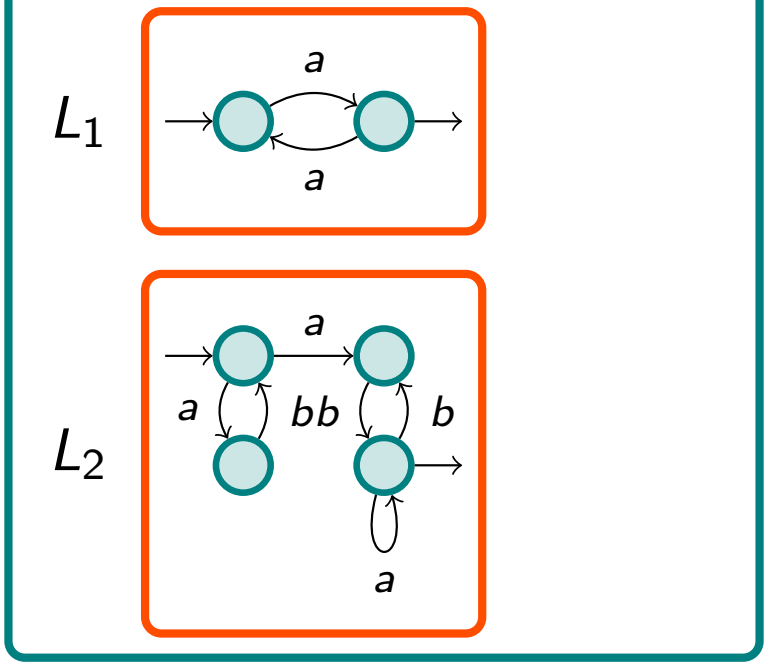
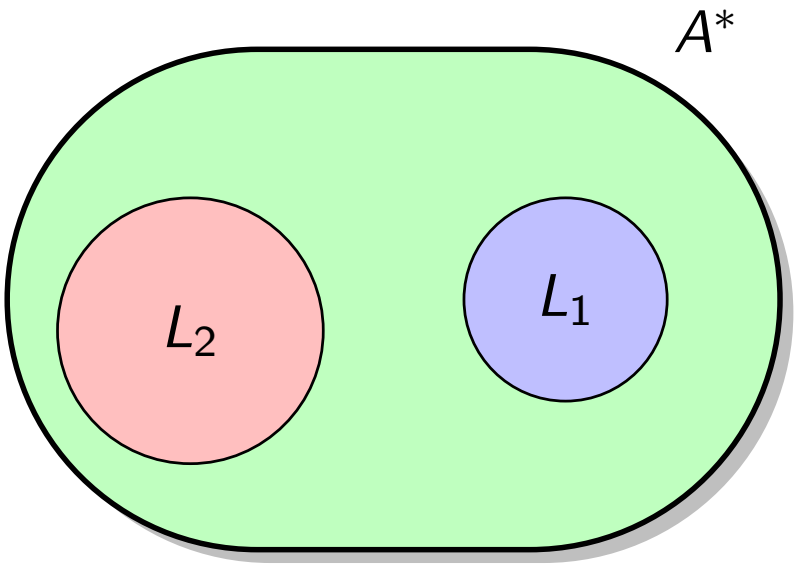


Can L_1 be separated from L_2 with a sentence of $\mathcal{C}$?



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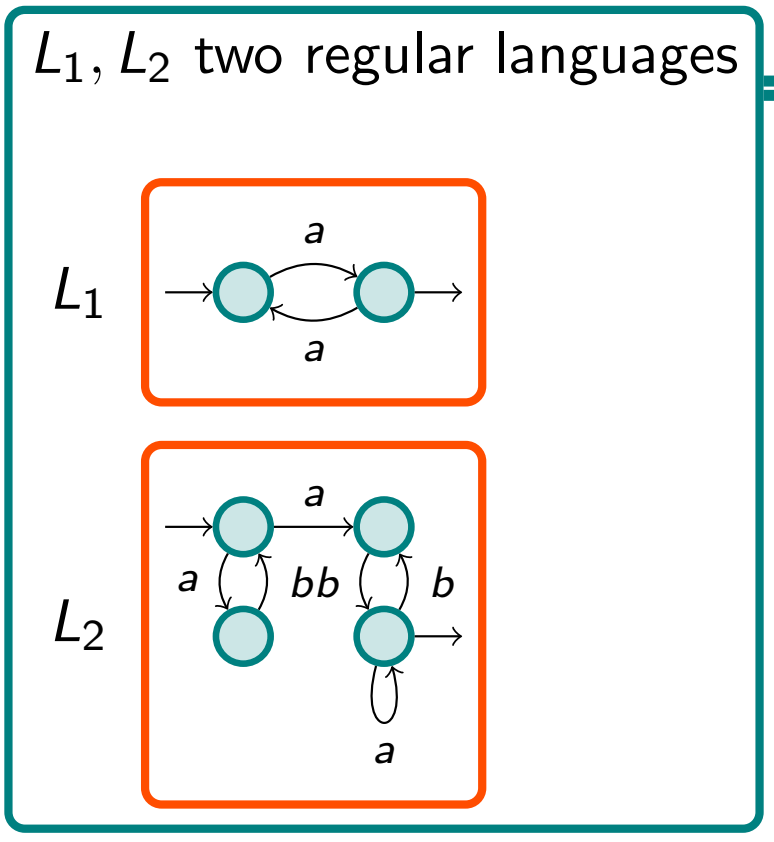

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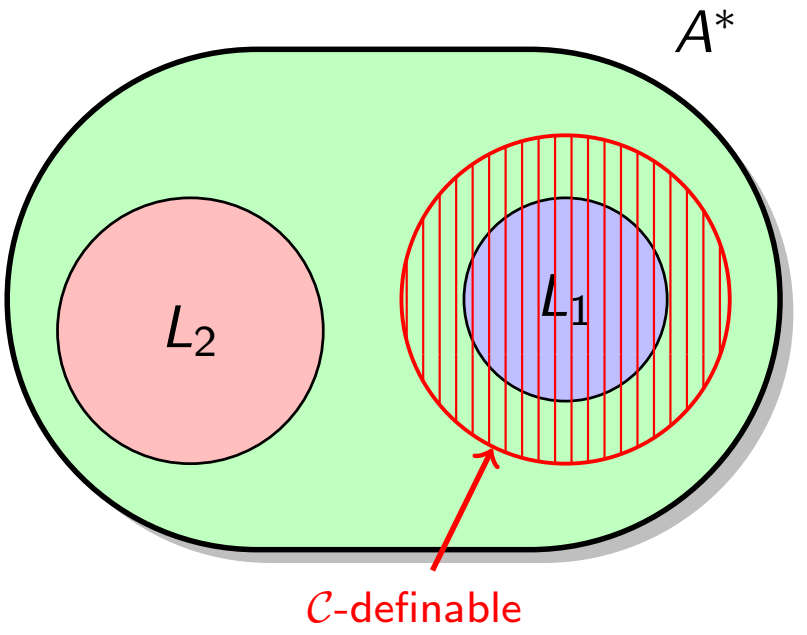

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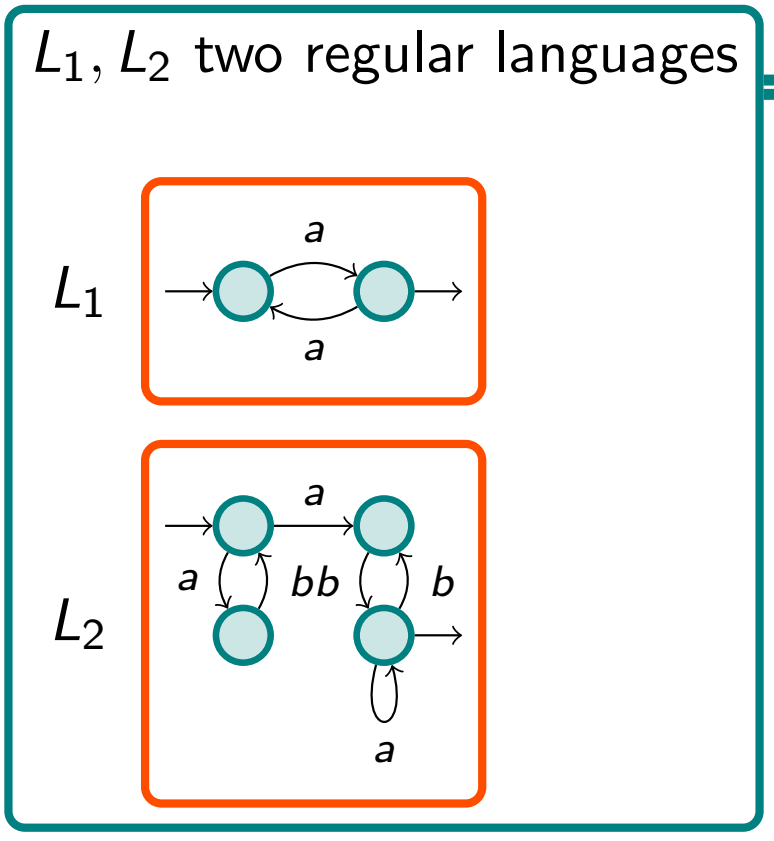

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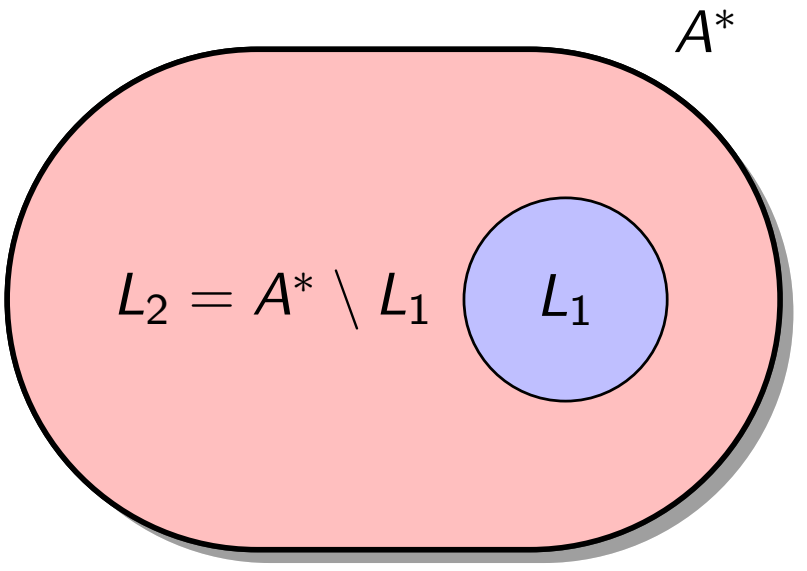

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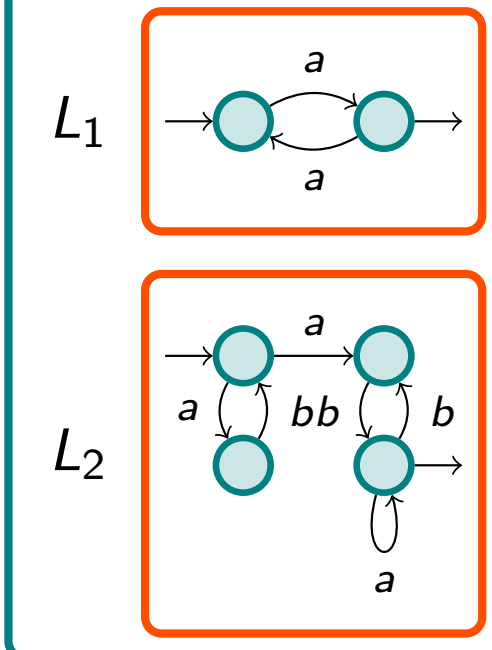

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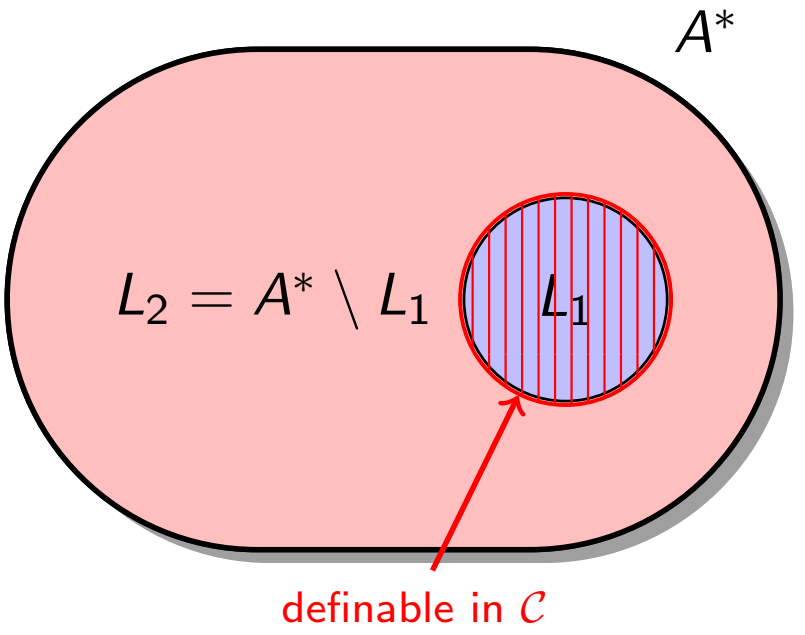

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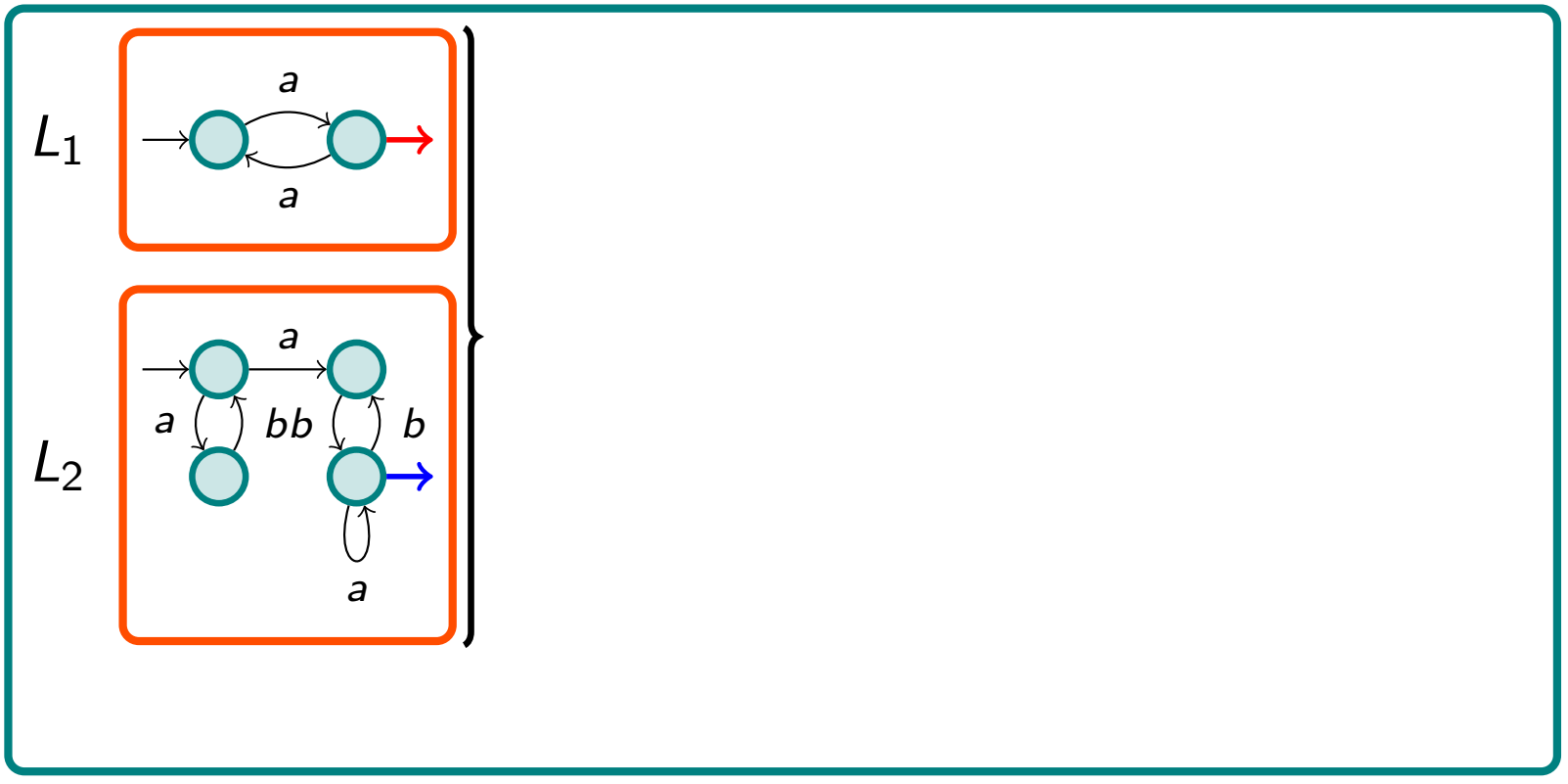
A more General Problem than Membership



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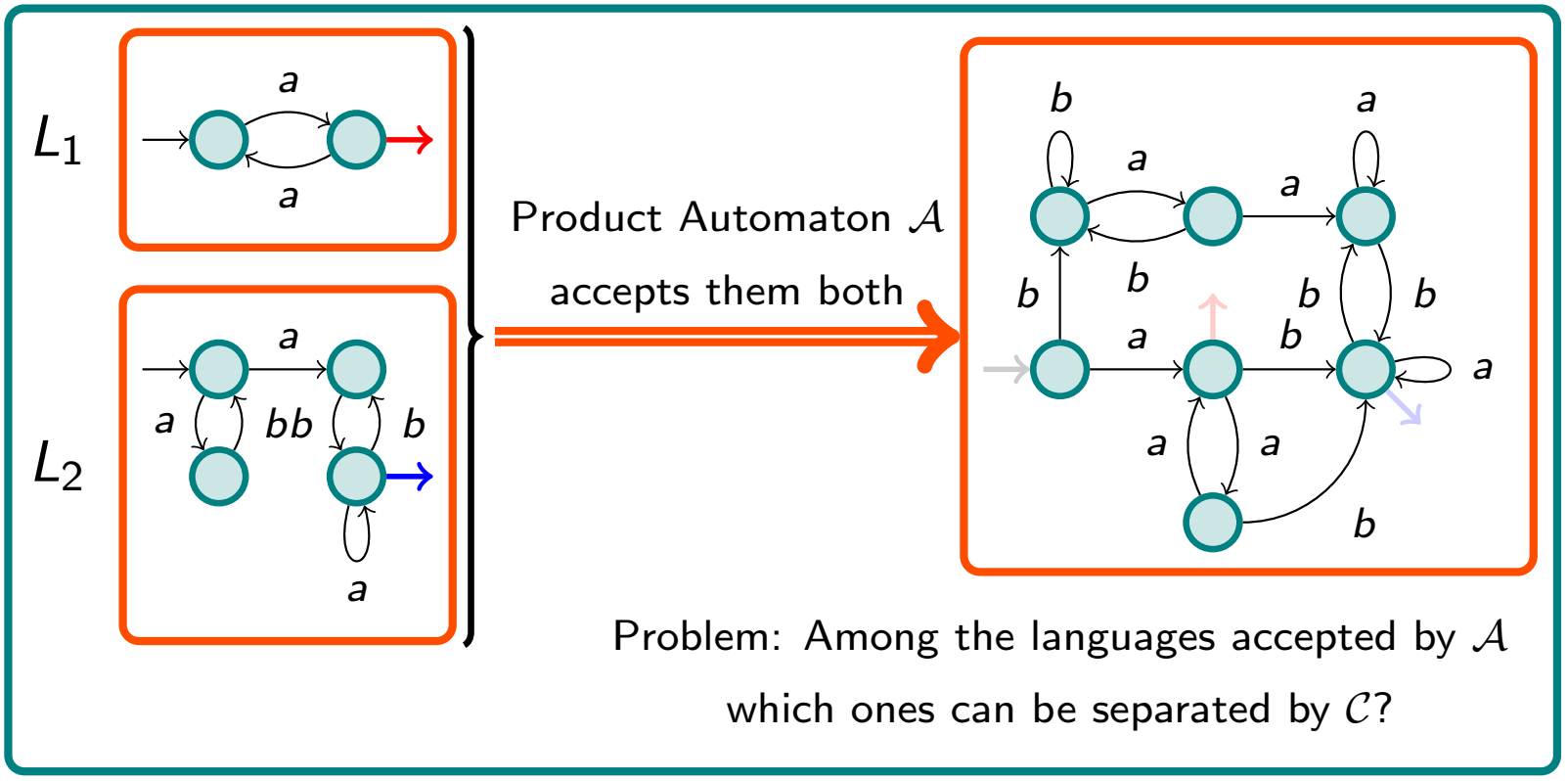

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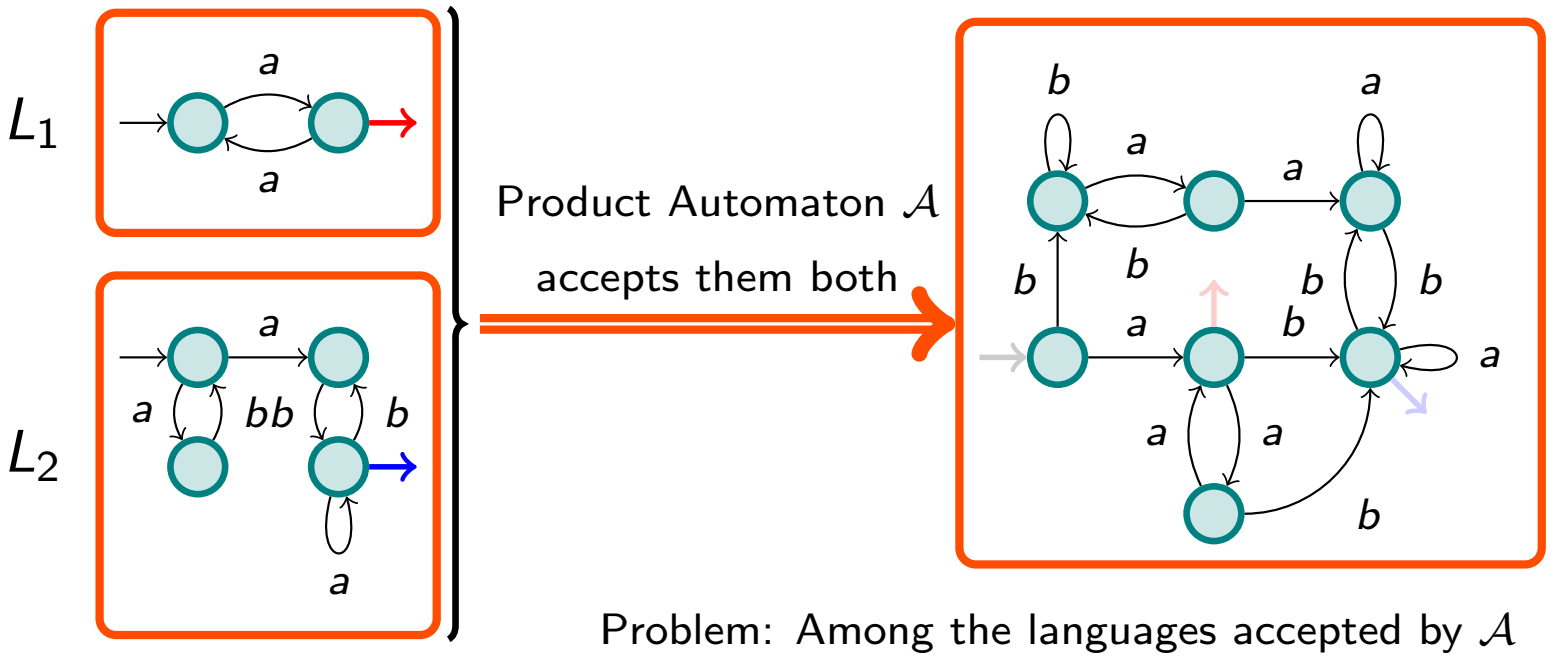

Problem: Among the languages accepted by $\mathcal{A}$ which ones can be separated by $\mathcal{C}$?

Membership

1) Can we define **everything** about the input with $\mathcal{C}$?

2) If yes, compute sentences.

Separation

1) How well can the languages be **approximated** with $\mathcal{C}$?

2) Compute sentences realizing this best approximation.

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3. Σ_{n+1} is now equivalent to a syntactic criterion on the syntactic which depends on the above pairs.

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Important Remark

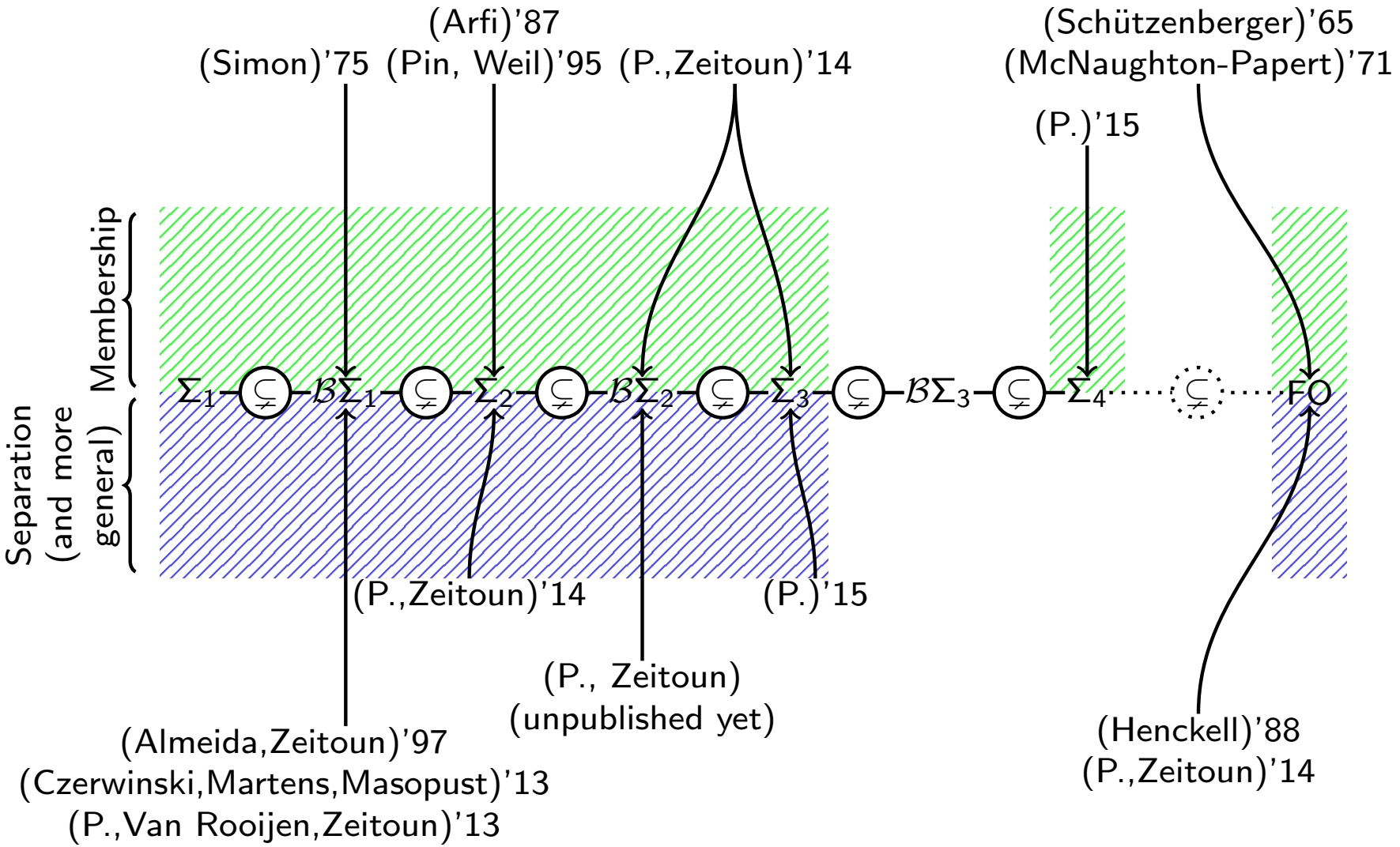
Separation is **harder** than membership. The theorem above does **not** solve the whole hierarchy.

The example of $\mathcal{B}\Sigma_2$

A similar theorem holds between Σ_2 and $\mathcal{B}\Sigma_2$. However,

1. the theorem is specific to level $n = 2$.
2. the connection is based on a problem **even more general** than separation (which we call "pointed covering").

Current state of the art: Approximation Problems



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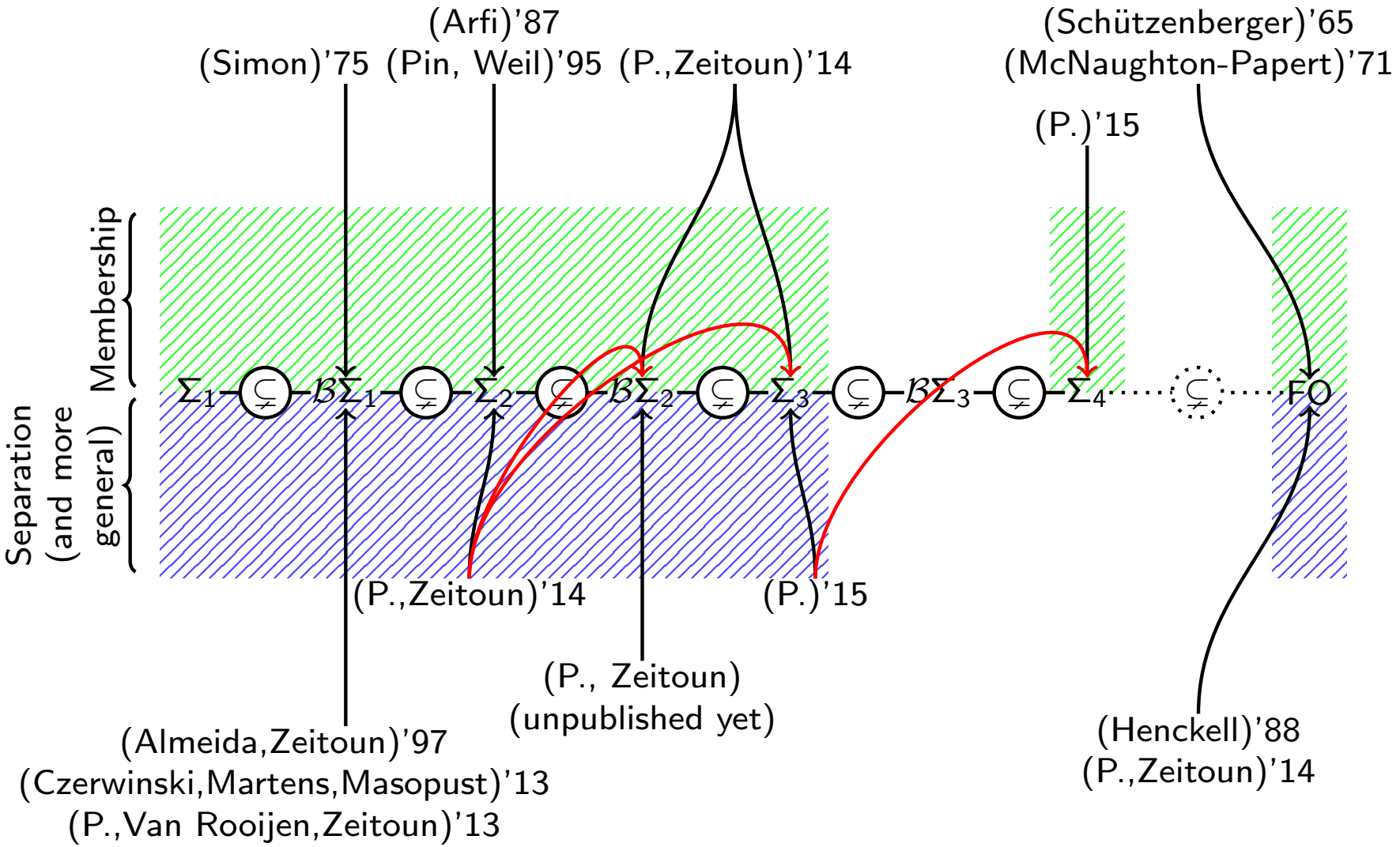

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The big problem right now:

No transfer result seems possible for approximation problems

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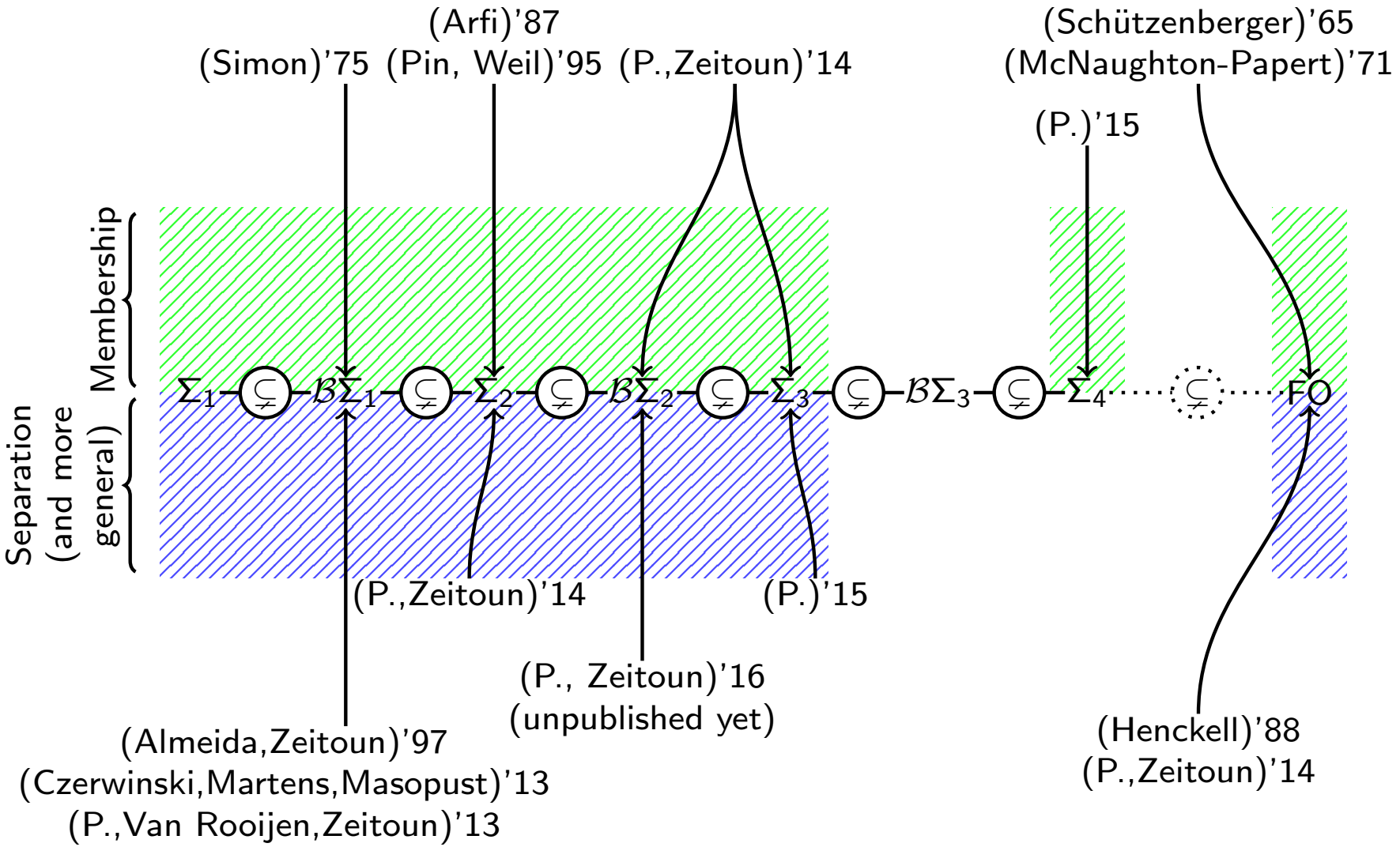
(Arfi)'87
(Simon)'75
(Pin, Weil)'95
(P.,Zeitoun)'14
(Schützenberger)'65
(McNaughton-Papert)'71
(P.)'15
Membership
Separation
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Σ_1
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FO
(P.,Zeitoun)'14
(P.)'15
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(Almeida,Zeitoun)'97
(Czerwinski,Martens,Masopust)'13
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(Henckell)'88
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Thank You