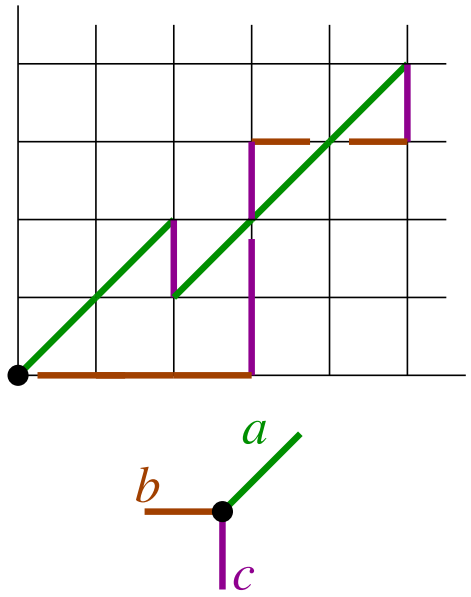
Lattice walks and algebraic power series

Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux



Le memoire presenté par Mr Bousquet-Melou " q-enumeration de polyominoaux convexes est un bel exemple des Travaux de l'ecole Bordelais de combinatoire.

Les objets etudiés constituent un cadre naturel pour examiner certains cas particuliers importants déja rencontrés par d'autres chercheurs. Les methodes employées vont de la construction d'objets planaires facilitant l'intuition geometrie aux diverses techniques algebriques de la combinatoire. Certains de ces methods (les q-automate) sont tout a fait originales et me paraissent promises à un avenir certain. Parmi les resultats obtenus, il faut relever la resolution de plusieurs conjectures difficiles liées à la percolation, d'autres indiquent des voies interessantes pour des recherches futures (la positivité du numerateur de certaines series fractionnelles).

D'un bout à l'autre du travail le lecteur appreciera la clarté et la rigueur de l'exposé et un très grand talent pour presenter des constructions ingenieuses souvent de facture.

Faisant en outre preuve d'une large connaissance des recherches combinatoires contemporaines et classiques, le memoire de Mr Bousquet-Melou me paraît etre une très bonne These de mathematiques

M. Schützenberger

I. The Schützenberger methodology

Enumerative combinatorics and generating functions

- Let $\mathcal{W}$ be a set of combinatorial objects equipped with an integer size $|\cdot|$, and assume that for each n, the set

$$\{w \in \mathcal{W} : |w| = n\},$$

is finite. Let $c(n)$ be its cardinality.

- The generating function of the objects of $\mathcal{W}$, counted by the size, is

$$\mathrm{C}(t) := \sum_{n \geq 0} c(n) t^n = \sum_{w \in \mathcal{W}} t^{|w|}.$$

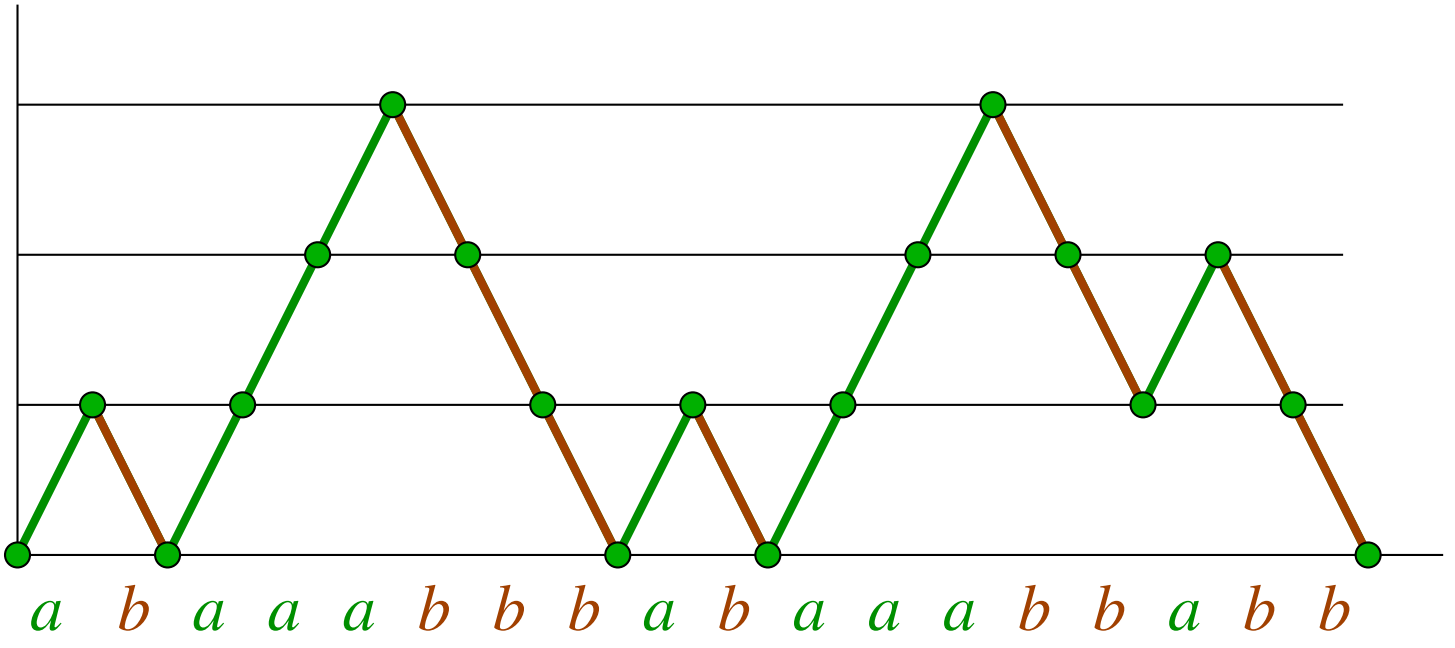
Also called "generating function of the sequence $c(n)$".

Example: the Dyck language

• Let $\mathcal{D}$ be the Dyck language on $\{a, b\}$, generated by the context-free grammar

$$D \to \varepsilon, \qquad D \to aDbD.$$

Then $\mathcal{D} = \{\varepsilon, ab, abab, aabb, \ldots\}$ consists of words in which every prefix contains at least as many a's as b's, and the whole word exactly as many a's as b's. Graphically:



Example: the Dyck language

• Let $\mathcal{D}$ be the Dyck language on $\{a, b\}$, generated by the context-free grammar

$$D \to \varepsilon, \qquad D \to aDbD.$$

Then $\mathcal{D} = \{\varepsilon, ab, abab, aabb, \ldots\}$ consists of words in which every prefix contains at least as many a's as b's, and the whole word exactly as many a's as b's.

• The generating function of Dyck words, counted by their length (number of letters) is:

$$\mathrm{D}(t) = \sum_{w \in \mathcal{D}} t^{|w|} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^{2n} = \frac{1 - \sqrt{1 - 4t^2}}{2t^2}.$$

This is an algebraic series.

Algebraic series

Definition. A formal power series $C(t)$ with coefficients in $\mathbb{Q}$ is algebraic if there exists a non-zero bivariate polynomial P with coefficients in $\mathbb{Q}$ such that

$$P(t, C(t)) = 0.$$

Algebraic series: the multivariate case

Definition. A formal power series $\mathsf{C}(t)$ with coefficients in $\mathbb{Q}$ is algebraic if there exists a non-zero bivariate polynomial P with coefficients in $\mathbb{Q}$ such that

$$P(t, \mathsf{C}(t)) = 0.$$

Multi-variate generating functions: for objects equipped with several parameters $p_1, \ldots, p_k$:

$$\mathsf{C}(x_1, \ldots, x_k; t) = \sum_{w \in \mathcal{W}} x_1^{p_1(w)} \cdots x_k^{p_k(w)} t^{|w|},$$

algebraicity means that

$$P(x_1, \ldots, x_k, t, \mathsf{C}(x_1, \ldots, x_k; t)) = 0$$

for some non-zero polynomial P with coefficients in $\mathbb{Q}$.

The Chomsky-Schützenberger theorem

Theorem (1963)

The generating function of words w of a non-ambiguous context-free language $\mathcal{W}$ on the alphabet $\mathcal{A}$, counted by the length $|w|$ (and possibly by the number $|w|_a$ of occurrences of each letter $a \in \mathcal{A}$), is algebraic.

Any non-ambiguous grammar generating $\mathcal{W}$ gives a system of algebraic equations defining the associated GF. For instance, for the Dyck language,

$$D \to \varepsilon, \qquad D \to aDbD$$

gives

$$\mathrm{D}(t) = 1 + t^2 \mathrm{D}(t)^2.$$

These series are even $\mathbb{N}$-algebraic, that is, solutions of positive systems.

$\mathbb{N}$-algebraic series

- A polynomial system

$$C_i = P_i(t, C_0, C_1, \ldots, C_k), \qquad \text{for } i \in [\![0, k]\!]$$

is proper if $P_i(0, \ldots, 0) = 0$ (no constant term) and $[C_j]P_i(t, C_0, \ldots, C_k) = 0$ (no linear term C_j).

- A proper system has a unique power series solution $(C_0(t), \ldots, C_k(t))$ with $C_i(0) = 0$.

Example:

$$C_0 = 2t + C_0^2 + 4tC_1,$$
$$C_1 = tC_0 + C_0C_1^2.$$

$\mathbb{N}$-algebraic series

- A polynomial system

$$\mathsf{C}_i = P_i(t, \mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_k), \qquad \text{for } i \in [\![0, k]\!]$$

is proper if $P_i(0, \ldots, 0) = 0$ (no constant term) and $[\mathsf{C}_j]P_i(t, \mathsf{C}_0, \ldots, \mathsf{C}_k) = 0$ (no linear term C_j).

- A proper system has a unique power series solution $(\mathsf{C}_0(t), \ldots, \mathsf{C}_k(t))$ with $\mathsf{C}_i(0) = 0$.

- A series $\mathsf{C}(t)$ is $\mathbb{N}$-algebraic if it reads $c_0 + \mathsf{C}_0(t)$ where $c_0 \in \mathbb{N}$ and $(\mathsf{C}_0(t), \ldots, \mathsf{C}_k(t))$ is the power series solution of a proper system with coefficients in $\mathbb{N}$.

Example: $\mathsf{C}(t) = 4 + \mathsf{C}_0(t)$ with

$$\mathsf{C}_0 = 2t + \mathsf{C}_0^2 + 4t\mathsf{C}_1,$$
$$\mathsf{C}_1 = t\mathsf{C}_0 + \mathsf{C}_0\mathsf{C}_1^2.$$

The Chomsky-Schützenberger theorem

Theorem (1963)

The generating function of words w of a non-ambiguous context-free language $\mathcal{W}$ on the alphabet $\mathcal{A}$, counted by the length $|w|$ (and possibly by the number $|w|_a$ of occurrences of each letter $a \in \mathcal{A}$), is $\mathbb{N}$-algebraic.

and vice-versa! (apart from the constant term)

The Schützenberger methodology

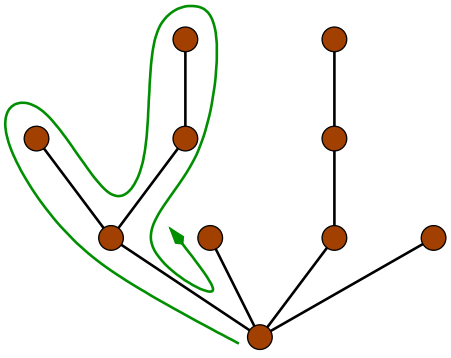
(Rough) idea

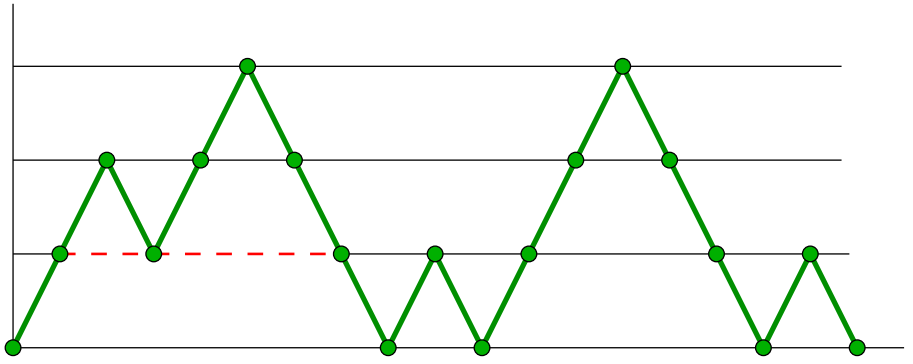
If a class of objects $\mathcal{O}$ has an algebraic generating function, one should try to explain this algebraicity via a size-preserving bijection between the objects of $\mathcal{O}$ and the words of a (non-ambiguous) context-free language.

The algebraic structure of words can be transfered into an algebraic structure on the objects themselves.

A basic example: plane trees

Plane trees with n edges are in bijection with Dyck words of length $2n$:

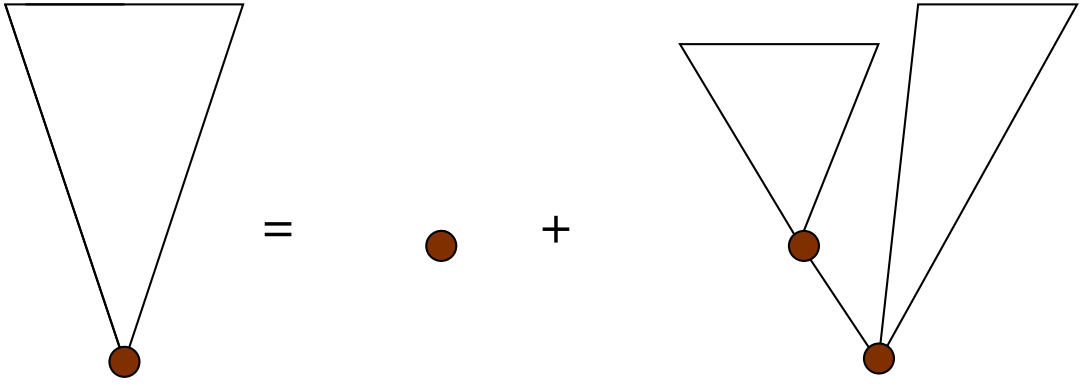




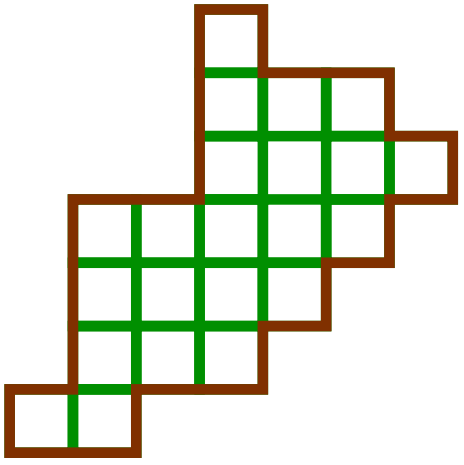
The context-free grammar

$$D \to \varepsilon, \qquad D \to aDbD$$

corresponds to the following recursive description of trees:

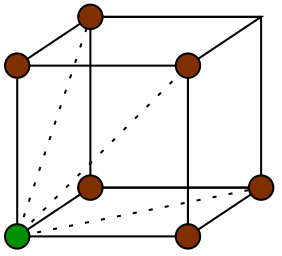


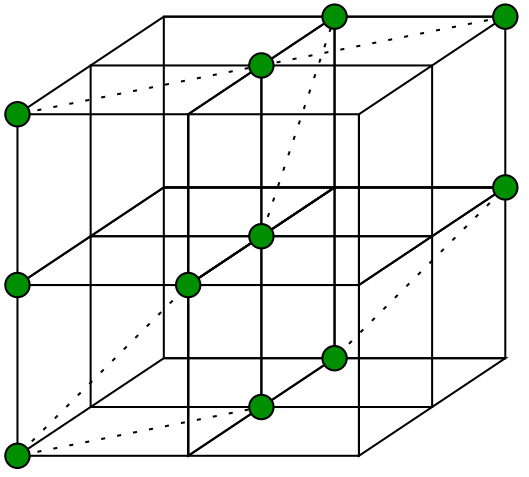
Growing up in the Schützenberger methodology



Directed convex polygons

$$\frac{t^4}{\sqrt{1-4t^2}}$$

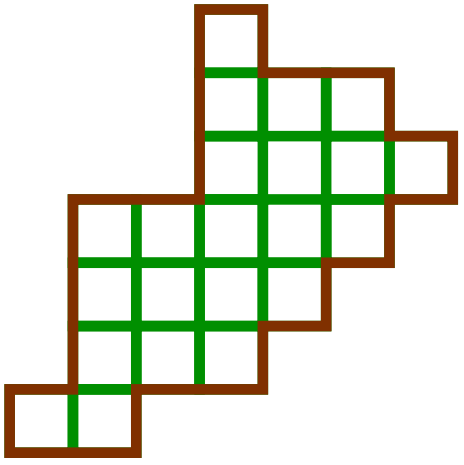




3D directed animals

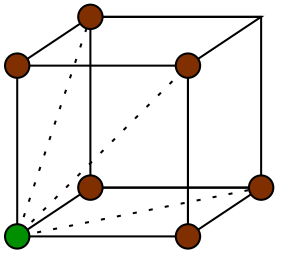
algebraic degree 12

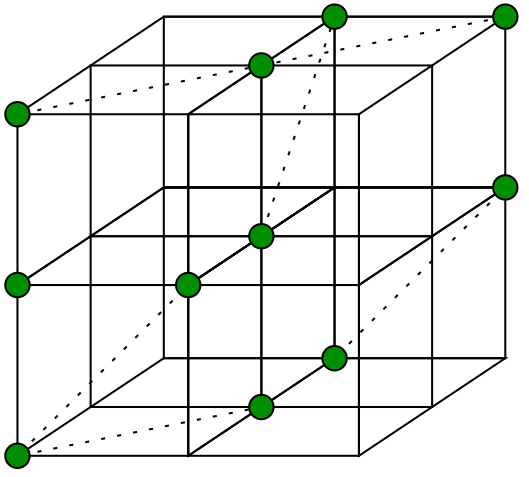
Growing up in the Schützenberger methodology



Directed convex polygons

$$\frac{t^4}{\sqrt{1-4t^2}}$$

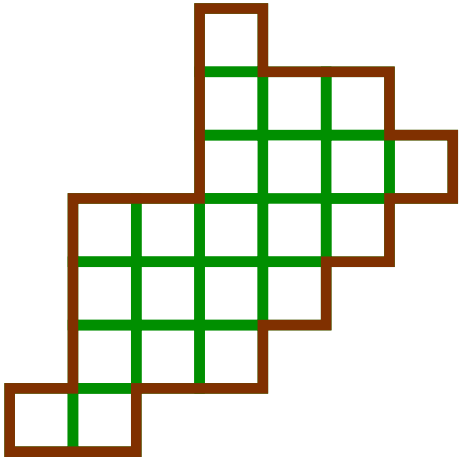




3D directed animals

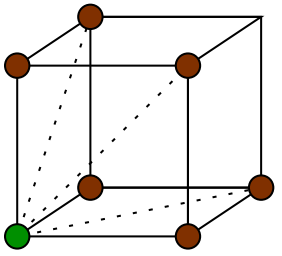
algebraic degree 12

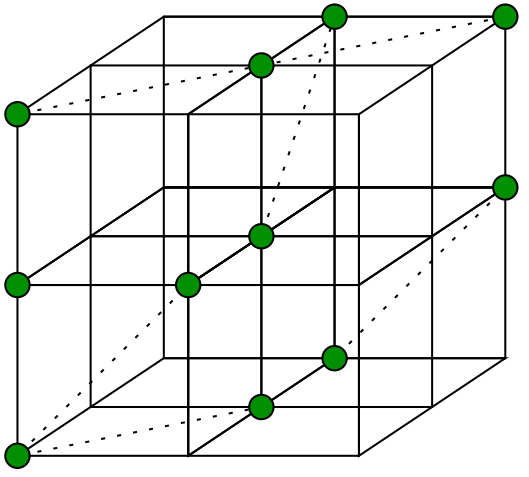
Growing up in the Schützenberger methodology



Directed convex polygons

$$\frac{t^4}{\sqrt{1-4t^2}}$$





3D directed animals

algebraic degree 12

In this talk: discuss the algebraicity of several enumeration problems, and their possible algebraic structure.
Recast them in a larger class of problems and series!

II. Three integer sequences

Three integer sequences

- Catalan numbers:

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

- Kreweras numbers:

$$K(n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$$

- Gessel numbers:

$$G(n) = 16^n \, \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$$

where $(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the ascending factorial.

What do they have in common?

They are hypergeometric

The ratio of two consecutive terms is rational in n:

$$\frac{C(n+1)}{C(n)} = 2\,\frac{2n+1}{n+2},$$

$$\frac{K(n+1)}{K(n)} = 6\,\frac{(3n+1)(3n+2)}{(n+2)(2n+3)},$$

$$\frac{G(n+1)}{G(n)} = 4\,\frac{(2n+1)(5n+6)}{(n+2)(3n+5)}.$$

The associated generating functions are algebraic

- Catalan generating function $\mathsf{C} \equiv \mathsf{C}(t)$:

$$\mathsf{C} = 1 + t\mathsf{C}^2,$$

- Kreweras generating function $\mathsf{K} \equiv \mathsf{K}(t)$:

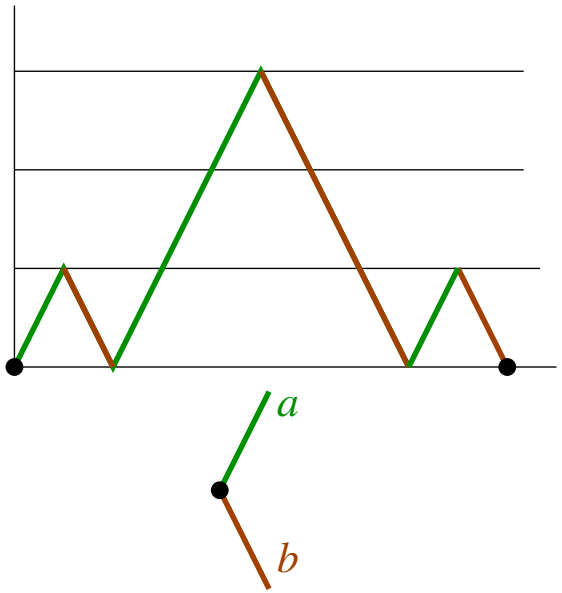
$$\mathsf{K} = 1 - 54t + 72t\mathsf{K} - 16t\mathsf{K}^2 - 64t^2\mathsf{K}^3,$$

- Gessel generating function $\mathsf{G} \equiv \mathsf{G}(t)$:

$$\begin{aligned}\mathsf{G} = 1 - 48t + 576t^2 + 256t^3 + 4t(15 - 228t + 128t^2)\mathsf{G} \\ + 2t(-5 + 156t - 312t^2 + 256t^3)\mathsf{G}^2 + 9t^2(-5 + 56t + 64t^2)\mathsf{G}^3 \\ + 9t^3(-13 + 28t + 32t^2)\mathsf{G}^4 - 189t^4\mathsf{G}^5 - 189t^5\mathsf{G}^6 \\ - 108t^6\mathsf{G}^7 - 27t^7\mathsf{G}^8.\end{aligned}$$

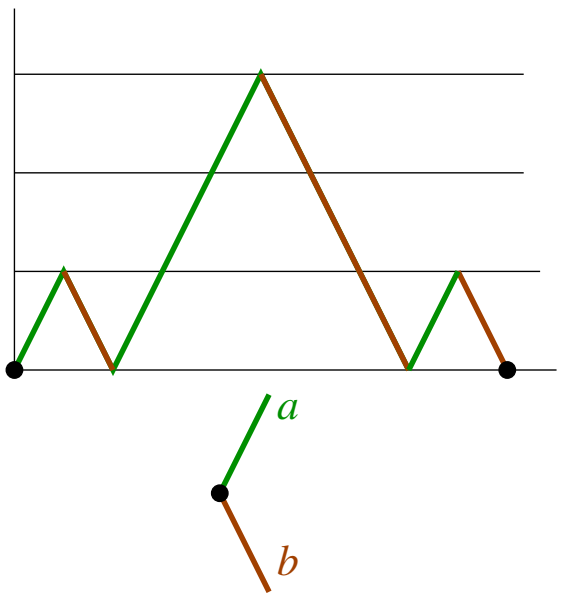
They count lattice paths (or Dyck-like words)

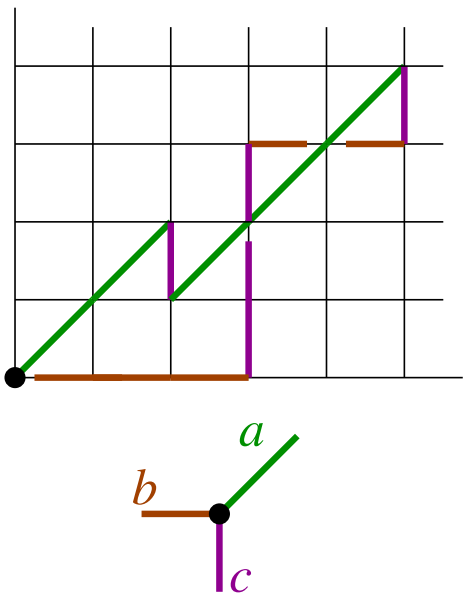
- Catalan $C(n)$ = number of words w on $\{a, b\}$ with $2n$ letters such that $|w|_a = |w|_b$ and for every prefix u of w, $|u|_a \geq |u|_b$.



They count lattice paths (or Dyck-like words)

- Catalan $C(n)$ = number of words w on $\{a, b\}$ with $2n$ letters such that $|w|_a = |w|_b$ and for every prefix u of w, $|u|_a \geq |u|_b$.
- Kreweras $K(n)$ = number of words w on $\{a, b, c\}$ with $3n$ letters such that $|w|_a = |w|_b = |w|_c$ and for every prefix u of w, $|u|_a \geq |u|_b$ and $|u|_a \geq |u|_c$. [Kreweras 65]



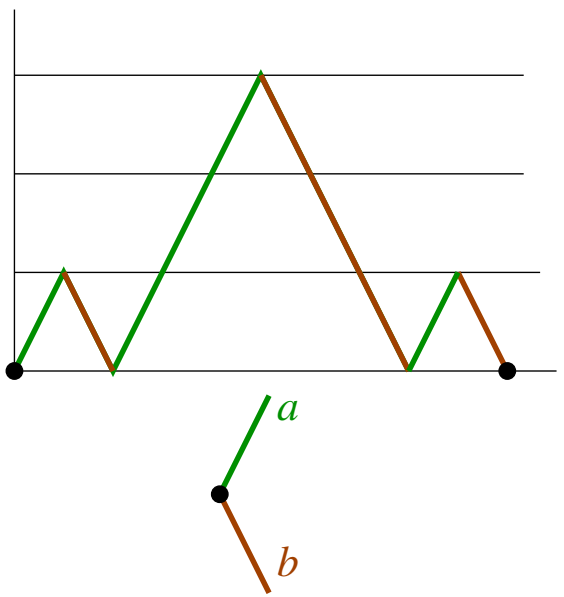


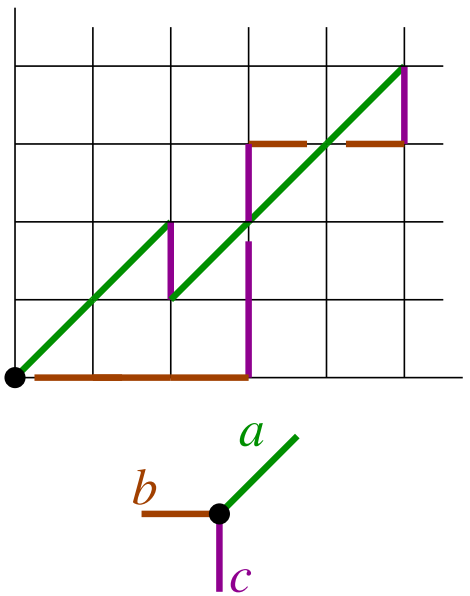
They count lattice paths (or Dyck-like words)

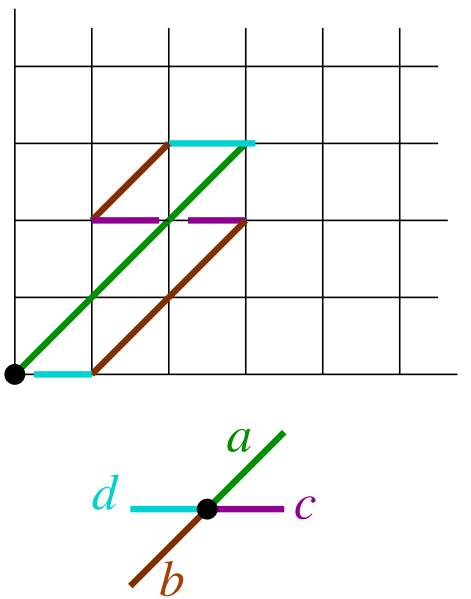
- Catalan $C(n)$ = number of words w on $\{a, b\}$ with $2n$ letters such that $|w|_a = |w|_b$ and for every prefix u of w, $|u|_a \geq |u|_b$.

- Kreweras $K(n)$ = number of words w on $\{a, b, c\}$ with $3n$ letters such that $|w|_a = |w|_b = |w|_c$ and for every prefix u of w, $|u|_a \geq |u|_b$ and $|u|_a \geq |u|_c$. [Kreweras 65]

- Gessel $G(n)$ = number of words w on $\{a, b, c, d\}$ with $4n$ letters such that $|w|_a = |w|_b = |w|_c = |w|_d$ and for every prefix u of w, $|u|_a \geq |u|_b$ and $|u|_a + |u|_c \geq |u|_b + |u|_d$. [Kauers et al. 09]







Some walks are (much) easier to count than others

Dyck walks [Whitworth 1878, Bertrand 1887, André 1887]

Some walks are (much) easier to count than others

Dyck walks [Whitworth 1878, Bertrand 1887, André 1887]

Kreweras walks

- first proof [Kreweras 65]
- simplifications [Niederhausen 80, 83], another approach [Gessel 86]
- first (and only) bijective proof [Bernardi 07]

Some walks are (much) easier to count than others

Dyck walks [Whitworth 1878, Bertrand 1887, André 1887]

Kreweras walks

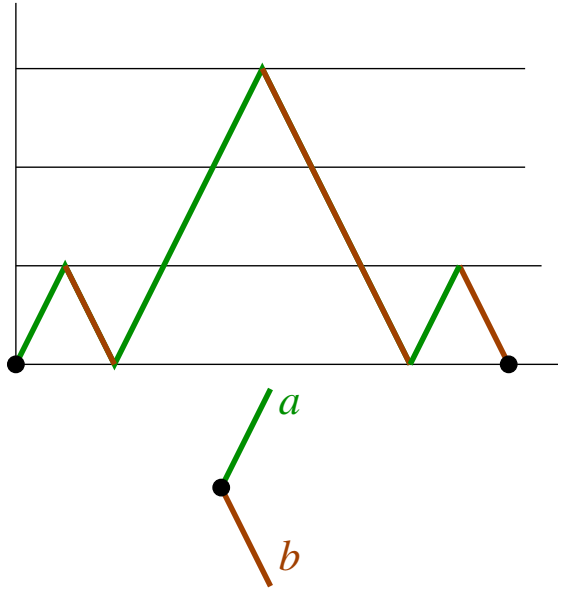
- first proof [Kreweras 65]
- simplifications [Niederhausen 80, 83], another approach [Gessel 86]
- first (and only) bijective proof [Bernardi 07]

Gessel walks

- conjecture [Gessel $\simeq$ 00]
- first proof for the numbers $G(n)$ by computer algebra [Kauers, Koutschan, Zeilberger 09]
- algebraicity [Bostan, Kauers 10] (computer algebra)
- "Human" proofs: [Bostan, Raschel 13(a)] (via complex analysis and elliptic functions) and [mbm 15(a)] (power series only).
- No bijective explanation.

III. Catalan numbers

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$



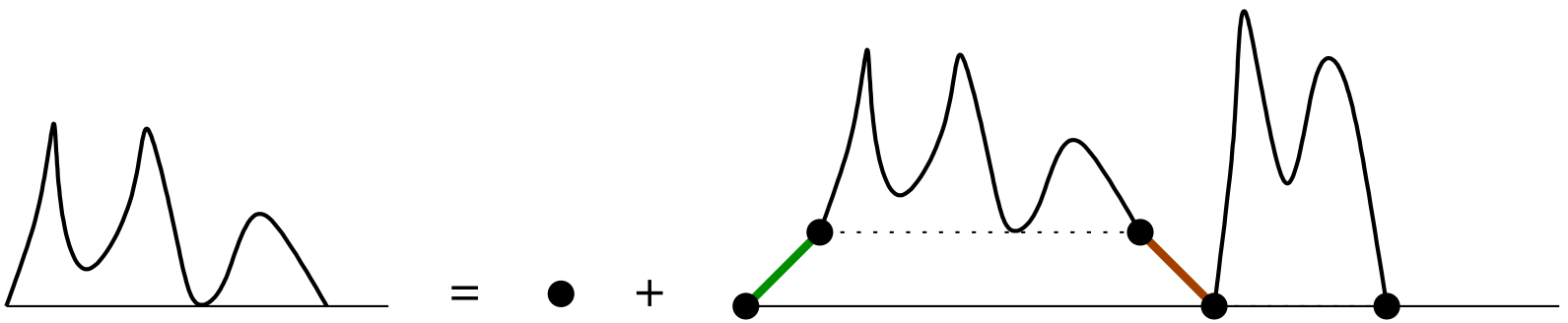
A ubiquitous quadratic structure

Quadratic recurrence relation:

$$C(n) = \sum_{i+j=n-1} C(i)C(j), \qquad C(0) = 1$$

Algebraic equation:

$$\mathrm{C}(t) = 1 + t\mathrm{C}(t)^2 \quad \Rightarrow \quad \mathrm{C}(t) = \frac{1-\sqrt{1-4t}}{2t}$$



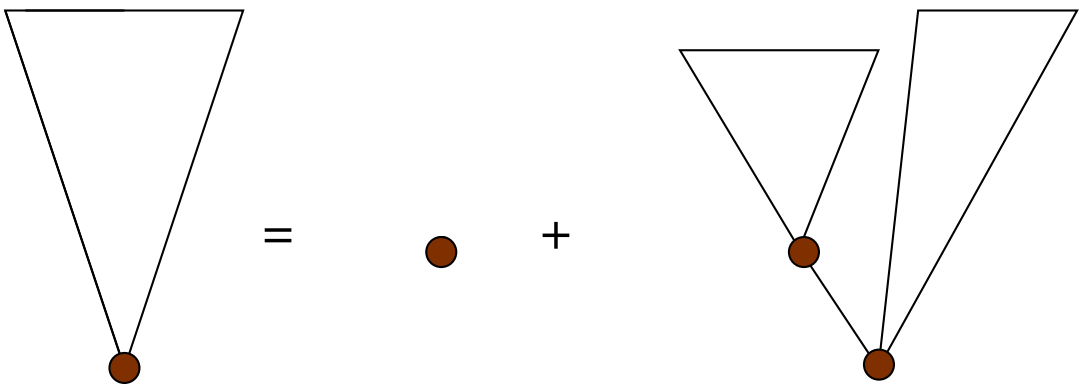
A ubiquitous quadratic structure

Quadratic recurrence relation:

$$C(n) = \sum_{i+j=n-1} C(i)C(j), \qquad C(0) = 1$$

Algebraic equation:

$$\mathrm{C}(t) = 1 + t\mathrm{C}(t)^2 \quad \Rightarrow \quad \mathrm{C}(t) = \frac{1 - \sqrt{1-4t}}{2t}$$



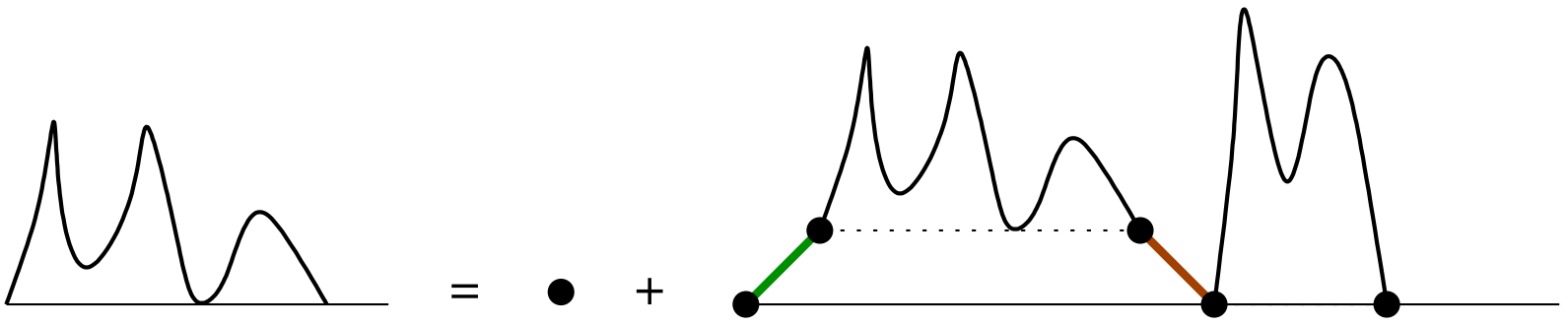
A ubiquitous quadratic structure

Quadratic recurrence relation:

$$C(n) = \sum_{i+j=n-1} C(i)C(j), \qquad C(0) = 1$$

Algebraic equation:

$$\mathrm{C}(t) = 1 + t\mathrm{C}(t)^2 \quad \Rightarrow \quad \mathrm{C}(t) = \frac{1-\sqrt{1-4t}}{2t}$$



Paths on a half-line are "context-free"

Proposition: for any finite step set $\mathcal{S} \subset \mathbb{Z}$, the set of walks going from 0 to 0 on the half-line $\mathbb{N}$ and taking their steps in $\mathcal{S}$ has an algebraic recursive description, obtained by first/last return decompositions.
[Labelle, Yeh 90, Labelle 93, Duchon 00]

Paths on a half-line are "context-free"

Proposition: for any finite step set $\mathcal{S} \subset \mathbb{Z}$, the set of walks going from 0 to 0 on the half-line $\mathbb{N}$ and taking their steps in $\mathcal{S}$ has an algebraic recursive description, obtained by first/last return decompositions.
[Labelle, Yeh 90, Labelle 93, Duchon 00]

Example: for $\mathcal{S} = \{3, -2\}$,

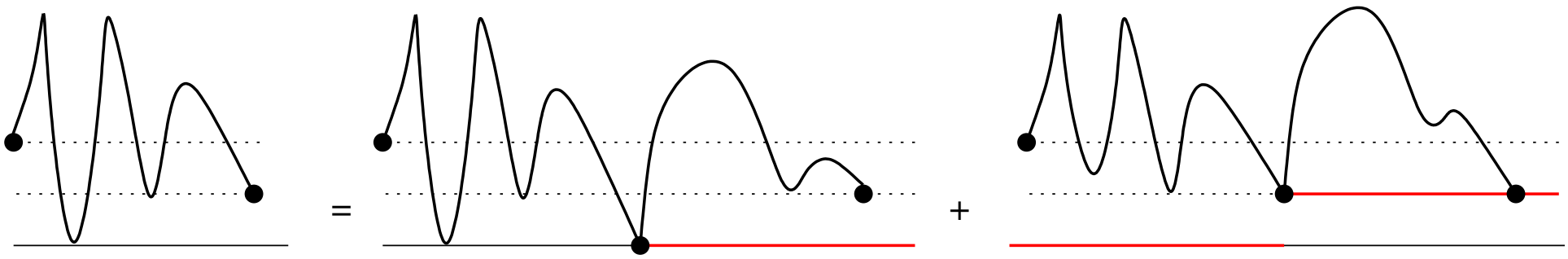
$$
\begin{aligned}
D_{0,0} &= 1 + C_{0,0}\, D_{0,0} \\
C_{0,0} &= a D_{2,1} b \\
D_{2,1} &= D_{2,0} C_{0,1} + D_{1,0} C_{0,0} \\
D_{1,0} &= C_{1,0}\, D_{0,0}
\end{aligned}
\qquad\qquad
\begin{aligned}
D_{2,0} &= C_{2,0}\, D_{0,0} \\
C_{1,0} &= D_{0,1}\, b \\
C_{2,0} &= D_{1,1}\, b \\
D_{1,1} &= D_{0,0} + D_{1,0}\, C_{0,1}
\end{aligned}
$$

Paths on a half-line are "context-free"

Proposition: for any finite step set $\mathcal{S} \subset \mathbb{Z}$, the set of walks going from 0 to 0 on the half-line $\mathbb{N}$ and taking their steps in $\mathcal{S}$ has an algebraic recursive description, obtained by first/last return decompositions.
[Labelle, Yeh 90, Labelle 93, Duchon 00]

Example: for $\mathcal{S} = \{3, -2\}$,

$$
\begin{aligned}
D_{0,0} &= 1 + C_{0,0}\, D_{0,0} & \qquad D_{2,0} &= C_{2,0}\, D_{0,0} \\
C_{0,0} &= a D_{2,1}\, b & \qquad C_{1,0} &= D_{0,1}\, b \\
D_{2,1} &= D_{2,0} C_{0,1} + D_{1,0} C_{0,0} & \qquad C_{2,0} &= D_{1,1}\, b \\
D_{1,0} &= C_{1,0}\, D_{0,0} & \qquad D_{1,1} &= D_{0,0} + D_{1,0}\, C_{0,1}
\end{aligned}
$$

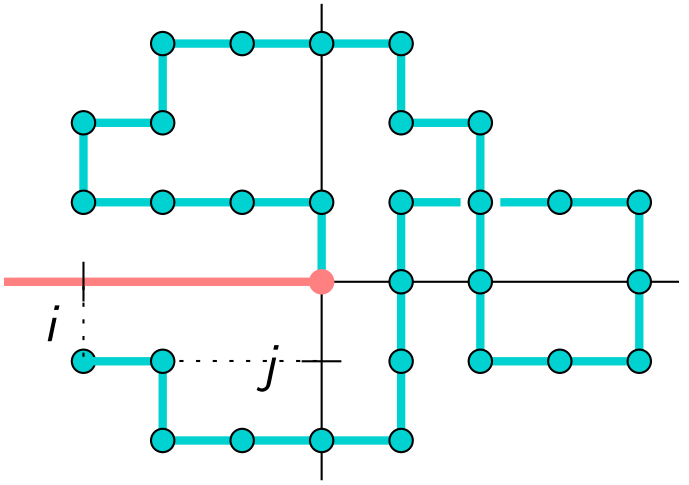


A Catalan mystery: Walks on the slit plane

Let $s(i,j;n)$ be the number of n-step walks on the square lattice that start at $(0,0)$ but never return to the non-positive x-axis, and end at (i,j). Then:

$$s(0,1;2n+1) = 4^n C(n),$$
$$s(1,0;2n+1) = C(2n+1),$$
$$s(-1,1;2n) = \frac{1}{2} C(2n).$$

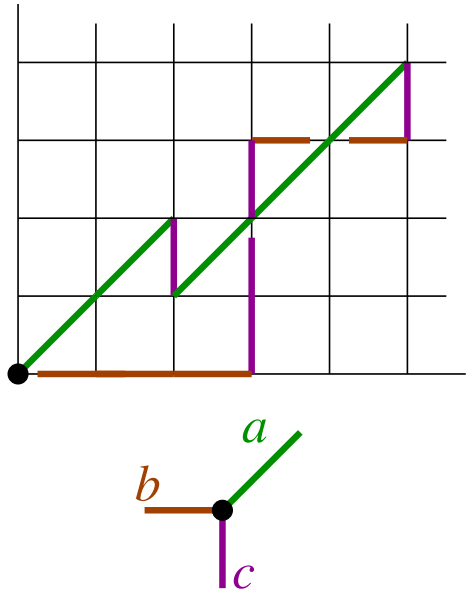
[mbm, Schaeffer 00(a)]



Bijective proof for $s(1,0;2n+1)$ [Barcucci, Pergola, Pinzani, Rinaldi 01]

IV. Kreweras numbers

$$K(n) = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n}$$



A negative result

Proposition: the GF of Kreweras numbers is NOT $\mathbb{N}$-algebraic, hence these walks do NOT have a positive algebraic recursive description.

A negative result

Proposition: the GF of Kreweras numbers is NOT $\mathbb{N}$-algebraic, hence these walks do NOT have a positive algebraic recursive description.

Proof:

$$K(n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n} \sim \frac{\sqrt{3}}{4\sqrt{\pi}}\, 27^n n^{-5/2}.$$

A negative result

Proposition: the GF of Kreweras numbers is NOT $\mathbb{N}$-algebraic, hence these walks do NOT have a positive algebraic recursive description.

Proof:

$$K(n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n} \sim \frac{\sqrt{3}}{4\sqrt{\pi}}\, 27^n n^{-5/2}.$$

Theorem [Banderier, Drmota 15]: If the n-th coefficient $a(n)$ of an $\mathbb{N}$-algebraic series satisfies

$$a(n) \sim c\mu^n n^{\alpha},$$

then $\alpha = -\frac{1}{2^k} - 1$, with $k \geq 1$ or $\alpha = \frac{m}{2^k} - 1$ with $m \in \{1, 2, 3, \ldots\}$ and $k \geq 0$. In particular, $\alpha \geq -3/2$.

A negative result

Proposition: the GF of Kreweras numbers is NOT $\mathbb{N}$-algebraic, hence these walks do NOT have a positive algebraic recursive description.

Proof:

$$K(n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n} \sim \frac{\sqrt{3}}{4\sqrt{\pi}}\, 27^n n^{-5/2}.$$

However, $K(t)$ is the difference of two $\mathbb{N}$-algebraic series of radius $1/27$:

$$K(t) = A(t) - 2tA(t)^4$$

with

$$A(t) = 1 + 4tA(t)^3 = \sum_{n\geq 0} \frac{4^n}{2n+1}\binom{3n}{n} t^n.$$

A negative result

Proposition: the GF of Kreweras numbers is NOT $\mathbb{N}$-algebraic, hence these walks do NOT have a positive algebraic recursive description.

Proof:

$$K(n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n} \sim \frac{\sqrt{3}}{4\sqrt{\pi}}\, 27^n n^{-5/2}.$$

However, $K(t)$ is the difference of two $\mathbb{N}$-algebraic series of radius $1/27$:

$$K(t) = A(t) - 2tA(t)^4$$

with

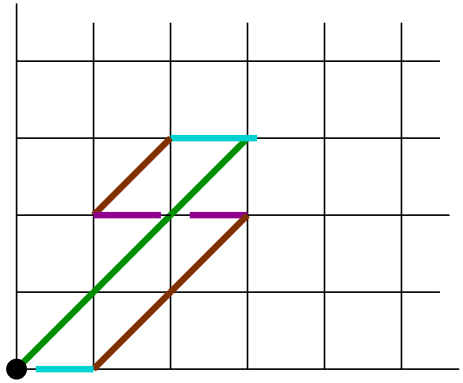
$$A(t) = 1 + 4tA(t)^3 = \sum_{n\geq 0} \frac{4^n}{2n+1}\binom{3n}{n} t^n.$$

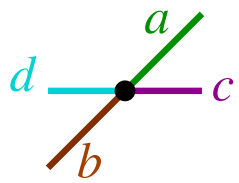
Combinatorial explanation of the algebraic system:

- a bijection with certain triangulations [Bernardi 07]...
- counted bijectively in [Poulalhon, Schaeffer 03].
- Many differences of $\mathbb{N}$-algebraic series in map enumeration [Cori, Vauquelin 80] $\Rightarrow$ Gilles Schaeffer's talk

V. Gessel numbers

$$G(n) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$$





A negative result

Proposition: the GF of Gessel numbers is NOT $\mathbb{N}$-algebraic, hence these walks do NOT have a positive algebraic recursive description.

Proof:

$$G(n) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} \sim c\, 16^n n^{-7/3}$$

Theorem [Banderier, Drmota 15] If the n-th coefficient $a(n)$ of an $\mathbb{N}$-algebraic series satisfies

$$a(n) \sim c \mu^n n^{\alpha},$$

then $\alpha = -\frac{1}{2^k} - 1$, with $k \geq 1$ or $\alpha = \frac{m}{2^k} - 1$ with $m \in \{1, 2, 3, \ldots\}$ and $k \geq 0$. In particular, $\alpha \geq -3/2$.

A negative result

Proposition: the GF of Gessel numbers is NOT $\mathbb{N}$-algebraic, hence these walks do NOT have a positive algebraic recursive description.

Proof:

$$G(n) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} \sim c\, 16^n n^{-7/3}$$

Theorem [Banderier, Drmota 15] If the n-th coefficient $a(n)$ of an $\mathbb{N}$-algebraic series satisfies

$$a(n) \sim c \mu^n n^{\alpha},$$

then $\alpha = -\frac{1}{2^k} - 1$, with $k \geq 1$ or $\alpha = \frac{m}{2^k} - 1$ with $m \in \{1, 2, 3, \ldots\}$ and $k \geq 0$. In particular, $\alpha \geq -3/2$.

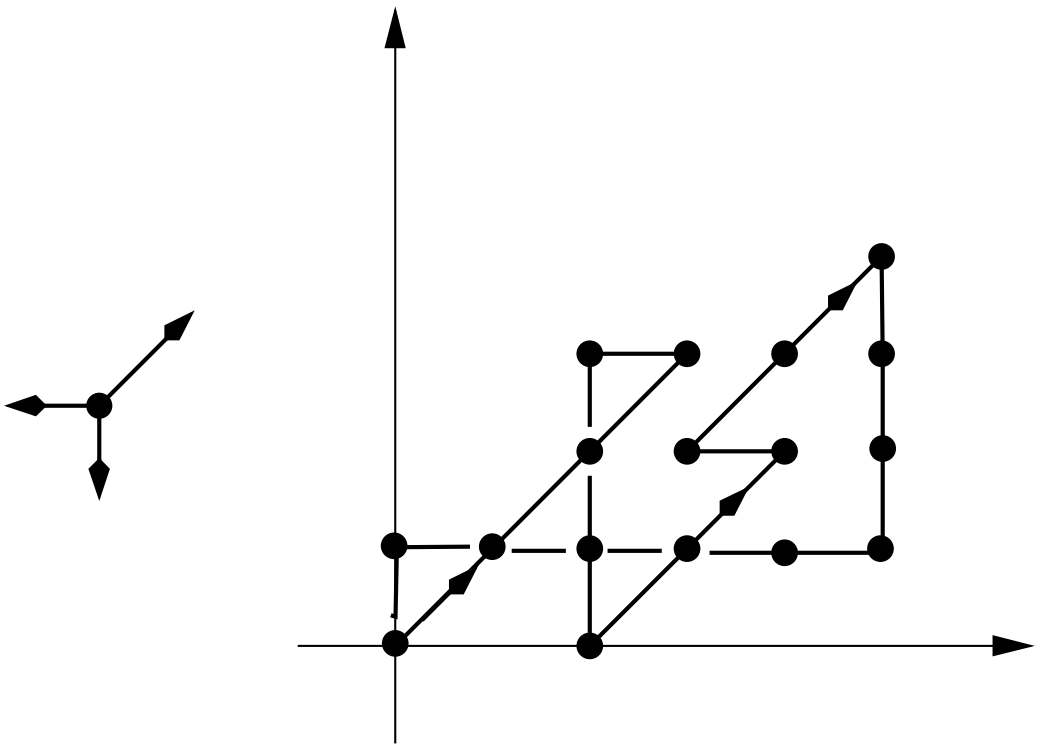
Questions

- Is G(t) is the difference of two $\mathbb{N}$-algebraic series of radius 1/16?
- Can the coefficients of an $\mathbb{N}$-algebraic series involve non-dyadic exponents in their asymptotic expansion?

VI. Where does algebraicity come from, then?

From other recursive descriptions,
and the corresponding (non-algebraic) functional equations

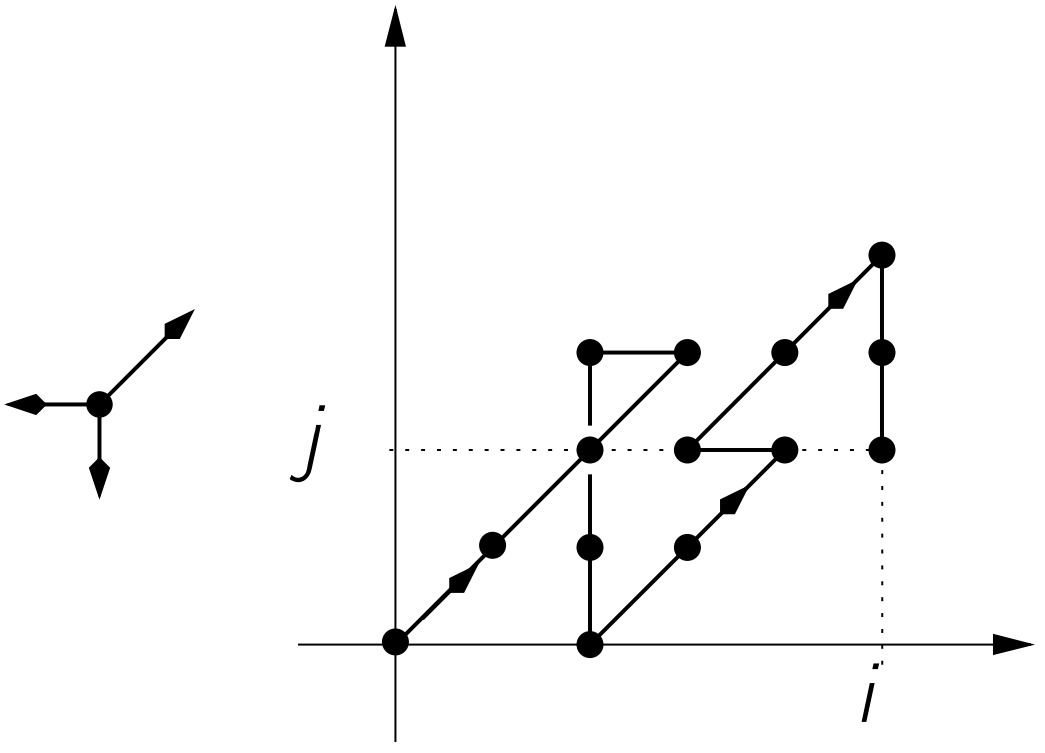
Kreweras walks: a step by step construction



Kreweras walks: a step by step construction

COST: consider Kreweras walks ending anywhere in the quadrant, and count them by the length and coordinates (i, j) of the endpoint:

$$K(x, y; t) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$



Kreweras walks: a step by step construction

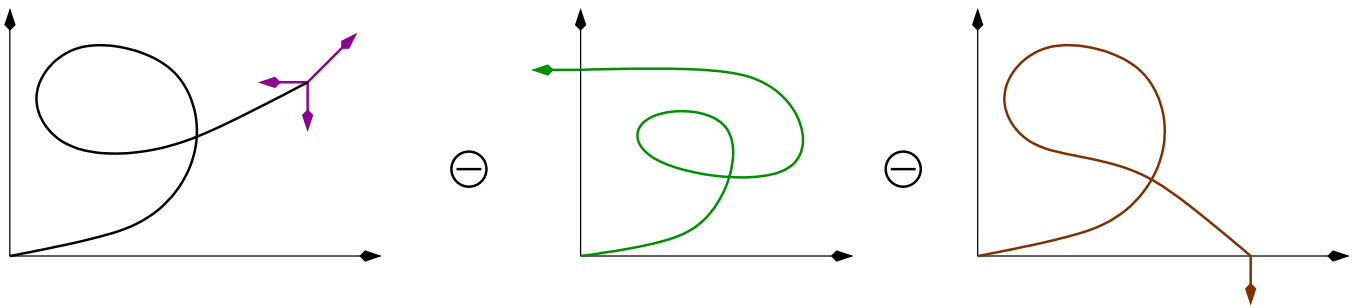
COST: consider Kreweras walks ending anywhere in the quadrant, and count them by the length and coordinates (i, j) of the endpoint:

$$K(x, y; t) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then

$$\boxed{K(x, y; t) \equiv K(x, y) = 1 + t(xy + \bar{x} + \bar{y})K(x, y) - t\bar{x}K(0, y) - t\bar{y}K(x, 0)}$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.



An equation with two catalytic variables [Zeilberger].

Kreweras walks: a step by step construction

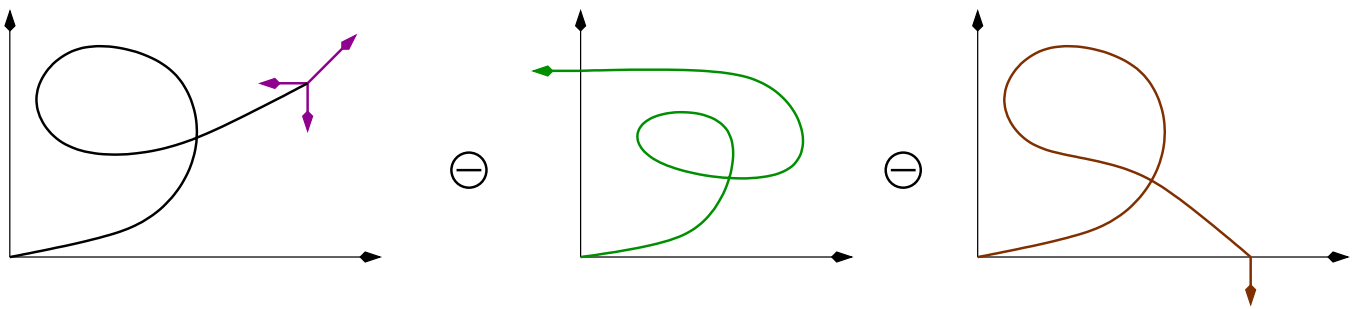
COST: consider Kreweras walks ending anywhere in the quadrant, and count them by the length and coordinates (i, j) of the endpoint:

$$K(x, y; t) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then

$$\boxed{K(x, y; t) \equiv K(x, y) = 1 + t(xy + \bar{x} + \bar{y})K(x, y) - t\bar{x}K(0, y) - t\bar{y}K(x, 0)}$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.



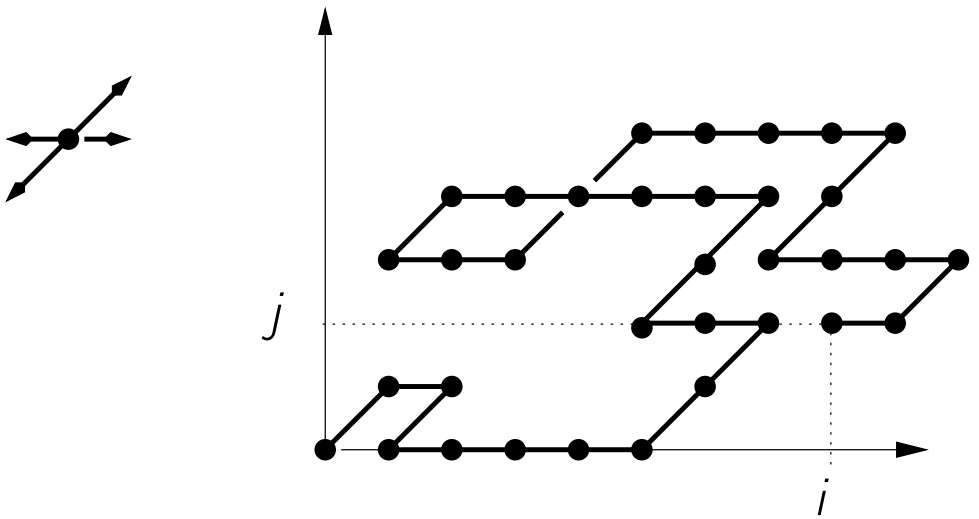
An equation with two catalytic variables [Zeilberger].

The series $K(x, y; t)$ is still algebraic [Gessel 86, mbm 02].

Gessel walks: a step by step construction

COST: consider Gessel walks ending anywhere in the quadrant, and count them by the length and coordinates (i, j) of the endpoint:

$$G(x, y; t) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$



Gessel walks: a step by step construction

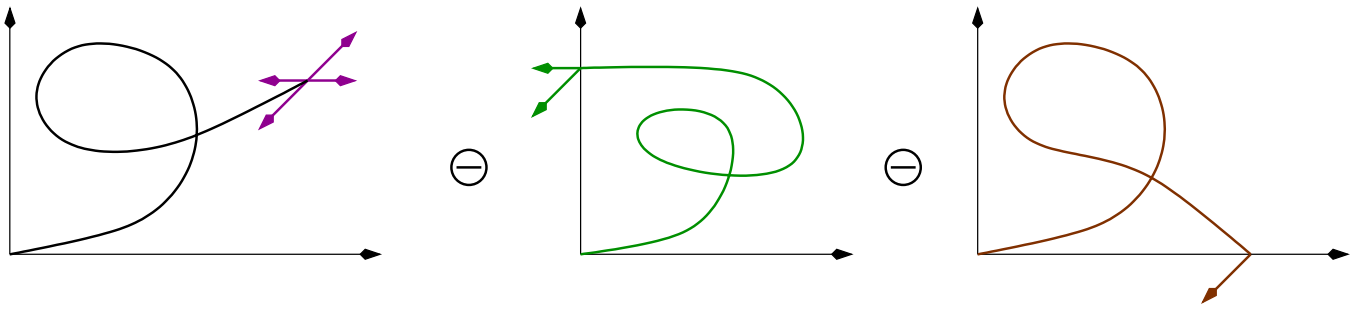
COST: consider Gessel walks ending anywhere in the quadrant, and count them by the length and coordinates (i, j) of the endpoint:

$$G(x, y; t) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then

$$G(x, y; t) \equiv G(x, y) = 1 + t(x + xy + \bar{x} + \bar{x}\bar{y})G(x, y) - t\bar{x}(1 + \bar{y})G(0, y) - t\bar{x}\bar{y}G(x, 0) + t\bar{x}\bar{y}G(0, 0),$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.

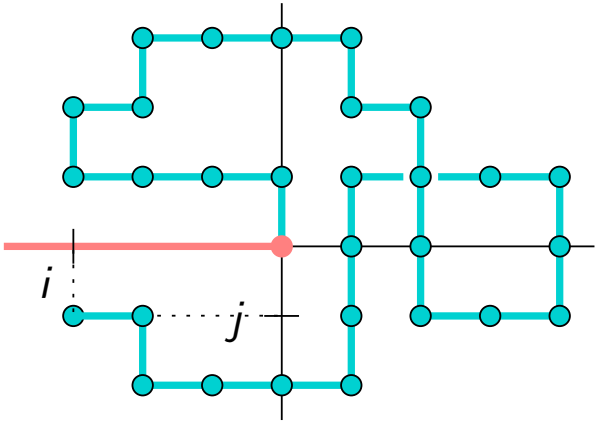


And... the series $G(x, y; t)$ is still algebraic [Bostan, Kauers 10].

Walks in the slit plane: a step by step construction

COST: consider square lattice walks avoiding the non-positive x-axis and ending anywhere, and count them by the length and coordinates (i, j) of the endpoint:

$$S(x, y; t) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$



Walks in the slit plane: a step by step construction

COST: consider square lattice walks avoiding the non-positive x-axis and ending anywhere, and count them by the length and coordinates (i, j) of the endpoint:

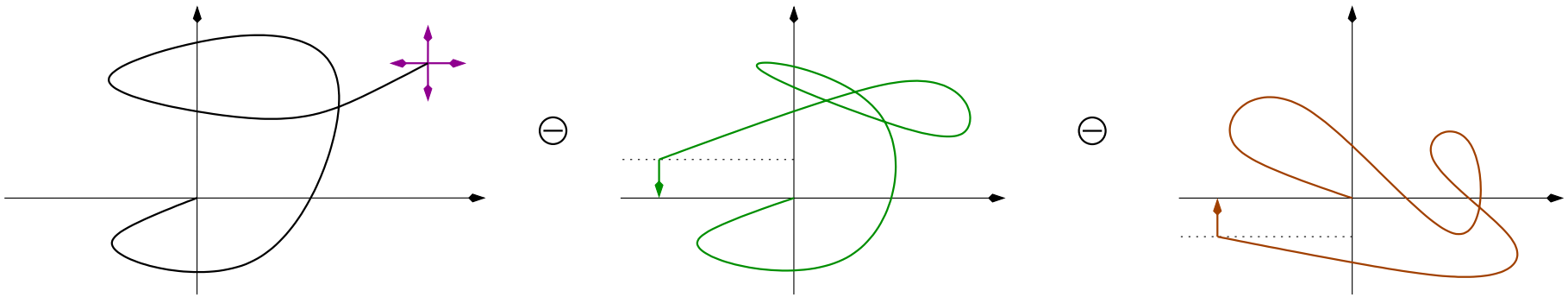
$$S(x, y; t) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then

$$S(x, y; t) \equiv S(x, y) = 1 + t(x + \bar{x} + y + \bar{y})S(x, y) - 2tS_1(x)$$

where

$$S_1(x) = \sum_{i \leq 0} s(i, 1; n) x^i t^n.$$

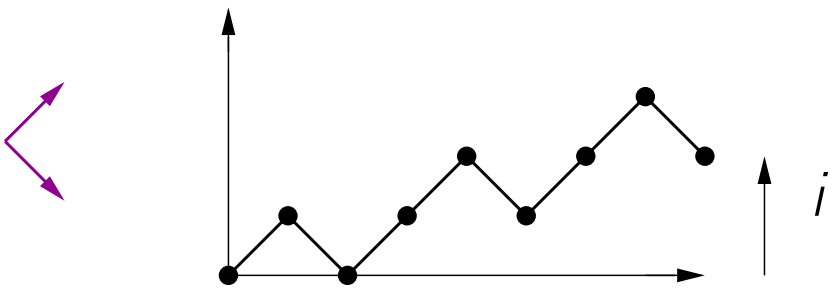


And... the series $S(x, y; t)$ is still algebraic [mbm, Schaeffer 00(a)].

Step by step for Dyck paths!

COST: consider walks with ± 1 steps on the non-negative half-line, and count them by the length and height i of the endpoint:

$$\mathrm{D}(x;t) = \sum_{w \ \text{walk}} t^{|w|} x^{i(w)}.$$



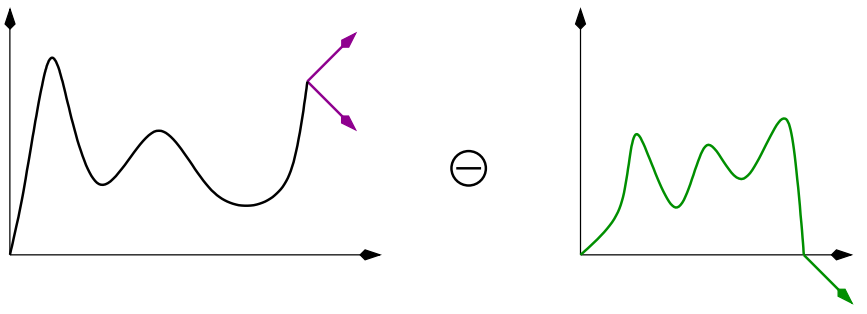
Step by step for Dyck paths!

COST: consider walks with ± 1 steps on the non-negative half-line, and count them by the length and height i of the endpoint:

$$\mathrm{D}(x;t) = \sum_{w \ walk} t^{|w|} x^{i(w)}.$$

Then

$$\mathrm{D}(x;t) \equiv \mathrm{D}(x) = 1 + t(x+\bar{x})\mathrm{D}(x) - t\bar{x}\mathrm{D}(0).$$



An equation with one catalytic variable.

The series $\mathrm{D}(x;t)$ is still algebraic (and even $\mathbb{N}$-algebraic).

Functional equations for 2D walks

Kreweras

$$K(x,y) = 1 + t(xy + \bar{x} + \bar{y})K(x,y) - t\bar{x}K(0,y) - t\bar{y}K(x,0)$$

Gessel

$$G(x,y) = 1 + t(x + xy + \bar{x} + \bar{x}\bar{y})G(x,y) - t\bar{x}(1+\bar{y})G(0,y) - t\bar{x}\bar{y}G(x,0) + t\bar{x}\bar{y}G(0,0)$$

Slit plane

$$S(x,y) = 1 + t(x + \bar{x} + y + \bar{y})S(x,y) - 2tS_1(x)$$

Why are these three series algebraic?

VII. Three more proofs of algebraicity for Dyck prefixes

$$\mathsf{D}(x) = 1 + t(x + \bar{x})\mathsf{D}(x) - t\bar{x}\mathsf{D}(0)$$

$$\Rightarrow \mathsf{D}(x) = 1 + 2\,xtD(x) + t\left(x^2 t + t - x\right)\mathsf{D}(x)^2.$$

Proof # 1: Guess and check

The equation (recall $\bar{x} = 1/x$):

$$\mathsf{D}(x) = 1 + t(x + \bar{x})\mathsf{D}(x) - t\bar{x}\mathsf{D}(0) \tag{1}$$

Proof # 1: Guess and check

The equation (recall $\bar{x} = 1/x$):

$$\mathrm{D}(x) = 1 + t(x + \bar{x})\mathrm{D}(x) - t\bar{x}\mathrm{D}(0) \tag{1}$$

- generate the first N coefficients of
  $\mathrm{D}(x; t) = 1 + tx + t^2(1 + x^2) + \cdots$ ($N = 10$ will suffice)

Proof # 1: Guess and check

The equation (recall $\bar{x} = 1/x$):

$$\mathrm{D}(x) = 1 + t(x + \bar{x})\mathrm{D}(x) - t\bar{x}\mathrm{D}(0) \tag{1}$$

- generate the first N coefficients of
  $\mathrm{D}(x;t) = 1 + tx + t^2(1 + x^2) + \cdots$ ($N = 10$ will suffice)
- guess a (conjectured) algebraic equation for D [Maple, GFUN]:

$$\mathrm{D}(x) = 1 + 2\,xt\mathrm{D}(x) + t\left(x^2t + t - x\right)\mathrm{D}(x)^2.$$

Proof # 1: Guess and check

The equation (recall $\bar{x} = 1/x$):

$$\mathrm{D}(x) = 1 + t(x + \bar{x})\mathrm{D}(x) - t\bar{x}\mathrm{D}(0) \tag{1}$$

- generate the first N coefficients of $\mathrm{D}(x;t) = 1 + tx + t^2(1 + x^2) + \cdots$ ($N = 10$ will suffice)
- guess a (conjectured) algebraic equation for D [Maple, GFUN]:

$$\mathrm{D}(x) = 1 + 2\,xt\mathrm{D}(x) + t\left(x^2t + t - x\right)\mathrm{D}(x)^2.$$

- one solution $\tilde{\mathrm{D}}(x)$ is a series in t with polynomial coefficients in x (so that $\tilde{\mathrm{D}}(0)$ is well-defined)

Proof # 1: Guess and check

The equation (recall $\bar{x} = 1/x$):

$$\mathrm{D}(x) = 1 + t(x + \bar{x})\mathrm{D}(x) - t\bar{x}\mathrm{D}(0) \tag{1}$$

- generate the first N coefficients of $\mathrm{D}(x;t) = 1 + tx + t^2(1 + x^2) + \cdots$ ($N = 10$ will suffice)
- guess a (conjectured) algebraic equation for D [Maple, GFUN]:

$$\mathrm{D}(x) = 1 + 2\,xt\mathrm{D}(x) + t\left(x^2t + t - x\right)\mathrm{D}(x)^2.$$

- one solution $\tilde{\mathrm{D}}(x)$ is a series in t with polynomial coefficients in x (so that $\tilde{\mathrm{D}}(0)$ is well-defined)
- check (1) for this conjectured value $\tilde{\mathrm{D}}(x)$.

Proof # 1: Guess and check

The equation (recall $\bar{x} = 1/x$):

$$\mathrm{D}(x) = 1 + t(x + \bar{x})\mathrm{D}(x) - t\bar{x}\mathrm{D}(0) \tag{1}$$

- generate the first N coefficients of $\mathrm{D}(x;t) = 1 + tx + t^2(1 + x^2) + \cdots$ ($N = 10$ will suffice)
- guess a (conjectured) algebraic equation for D [Maple, GFUN]:

$$\mathrm{D}(x) = 1 + 2\,xt\mathrm{D}(x) + t\left(x^2t + t - x\right)\mathrm{D}(x)^2.$$

- one solution $\tilde{\mathrm{D}}(x)$ is a series in t with polynomial coefficients in x (so that $\tilde{\mathrm{D}}(0)$ is well-defined)
- check (1) for this conjectured value $\tilde{\mathrm{D}}(x)$.

Applies (in principle) to many equations!

Proof # 1: Guess and check

The equation (recall $\bar{x} = 1/x$):

$$\mathrm{D}(x) = 1 + t(x + \bar{x})\mathrm{D}(x) - t\bar{x}\mathrm{D}(0) \tag{1}$$

- generate the first N coefficients of $\mathrm{D}(x;t) = 1 + tx + t^2(1 + x^2) + \cdots$ ($N = 10$ will suffice)
- guess a (conjectured) algebraic equation for D [Maple, GFUN]:

$$\mathrm{D}(x) = 1 + 2\,xt\mathrm{D}(x) + t\left(x^2t + t - x\right)\mathrm{D}(x)^2.$$

- one solution $\tilde{\mathrm{D}}(x)$ is a series in t with polynomial coefficients in x (so that $\tilde{\mathrm{D}}(0)$ is well-defined)
- check (1) for this conjectured value $\tilde{\mathrm{D}}(x)$.

Applies (in principle) to many equations!

$\Rightarrow$ One solution of Kreweras' and Gessel's problems [Bostan, Kauers 10]

Proof # 2: the kernel method

The equation:

$$(1 - t(x + \bar{x}))\, \mathrm{D}(x) = 1 - t\bar{x}\mathrm{D}(0)$$

- The kernel $(1 - t(x + \bar{x}))$, when solved for x, has two roots X and $1/X$, with
$$X = \frac{1 - \sqrt{1 - 4t^2}}{2t} = t + t^3 + O(t^5).$$
- Replacing x by X cancels the l.h.s., leaving
$$\mathrm{D}(0) = \frac{X}{t}.$$

[Knuth] The Art of Computer programming, Vol. 1 Chap. 2.

Applies to any problem of walks on the half-line $\mathbb{N}$ [Petkovšek 98].

Proof # 3: the algebraic kernel method

The equation:

$$(1 - t(x + \bar{x}))\, x\mathsf{D}(x) = x - t\mathsf{D}(0)$$

Proof # 3: the algebraic kernel method

The equation:

$$(1 - t(x + \bar{x}))\, x\mathrm{D}(x) = x - t\mathrm{D}(0)$$

- Replacing x by $1/x := \bar{x}$ leaves the kernel unchanged:

$$(1 - t(x + \bar{x}))\, \bar{x}\mathrm{D}(\bar{x}) = \bar{x} - t\mathrm{D}(0)$$

Proof # 3: the algebraic kernel method

The equation:

$$(1 - t(x + \bar{x}))\, x\mathrm{D}(x) = x - t\mathrm{D}(0)$$

- Replacing x by $1/x := \bar{x}$ leaves the kernel unchanged:

$$(1 - t(x + \bar{x}))\, \bar{x}\mathrm{D}(\bar{x}) = \bar{x} - t\mathrm{D}(0)$$

- Take the difference, and divide by the kernel:

$$x\mathrm{D}(x) - \bar{x}\mathrm{D}(\bar{x}) = \frac{x - \bar{x}}{1 - t(x + \bar{x})}$$

Proof # 3: the algebraic kernel method

The equation:

$$(1 - t(x + \bar{x}))\, xD(x) = x - tD(0)$$

- Replacing x by $1/x := \bar{x}$ leaves the kernel unchanged:

$$(1 - t(x + \bar{x}))\, \bar{x}D(\bar{x}) = \bar{x} - tD(0)$$

- Take the difference, and divide by the kernel:

$$xD(x) - \bar{x}D(\bar{x}) = \frac{x - \bar{x}}{1 - t(x + \bar{x})}$$

- Partial fraction expansion in x:

$$xD(x) - \bar{x}D(\bar{x}) = \frac{X}{t}\left(\frac{x}{1 - xX} - \frac{\bar{x}}{1 - \bar{x}X}\right)$$

Proof # 3: the algebraic kernel method

The equation:

$$(1 - t(x + \bar{x}))\, xD(x) = x - tD(0)$$

- Replacing x by $1/x := \bar{x}$ leaves the kernel unchanged:

$$(1 - t(x + \bar{x}))\, \bar{x}D(\bar{x}) = \bar{x} - tD(0)$$

- Take the difference, and divide by the kernel:

$$xD(x) - \bar{x}D(\bar{x}) = \frac{x - \bar{x}}{1 - t(x + \bar{x})}$$

- Partial fraction expansion in x:

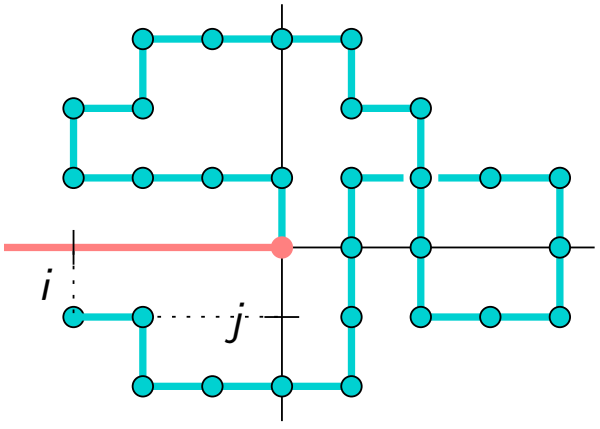
$$xD(x) - \bar{x}D(\bar{x}) = \frac{X}{t}\left(\frac{x}{1 - xX} - \frac{\bar{x}}{1 - \bar{x}X}\right)$$

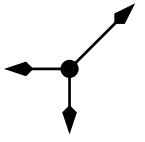
- Pick up positive powers of x:

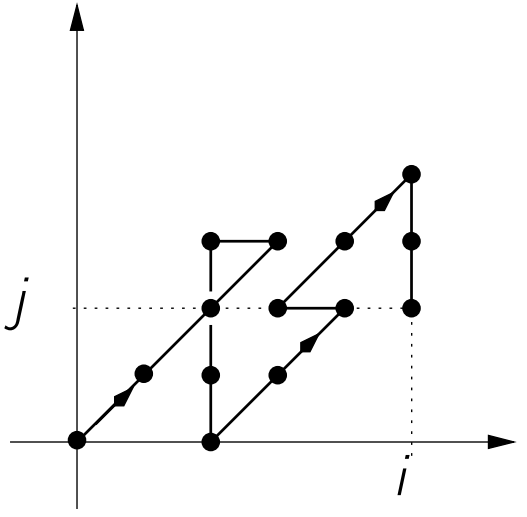
$$xD(x) = \frac{X}{t}\frac{x}{1 - xX}.$$

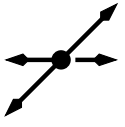
Two dimensional problems

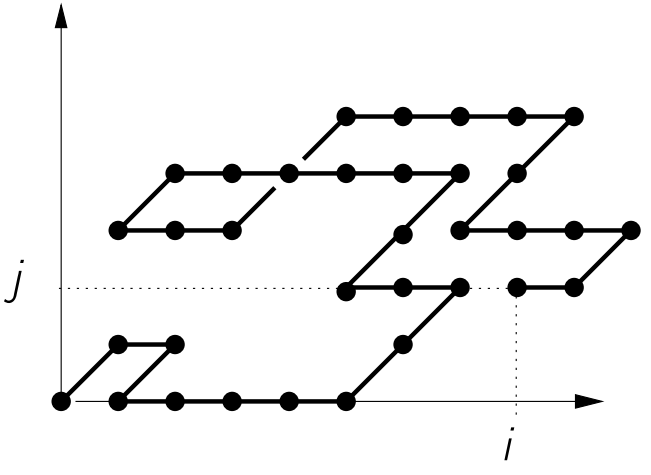
The above ideas (guess and check/ kernel method/ normal forms... and more) are used in the solutions of our 2D problems:



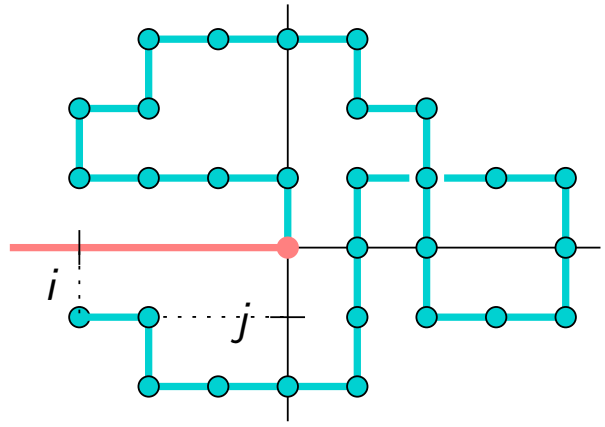




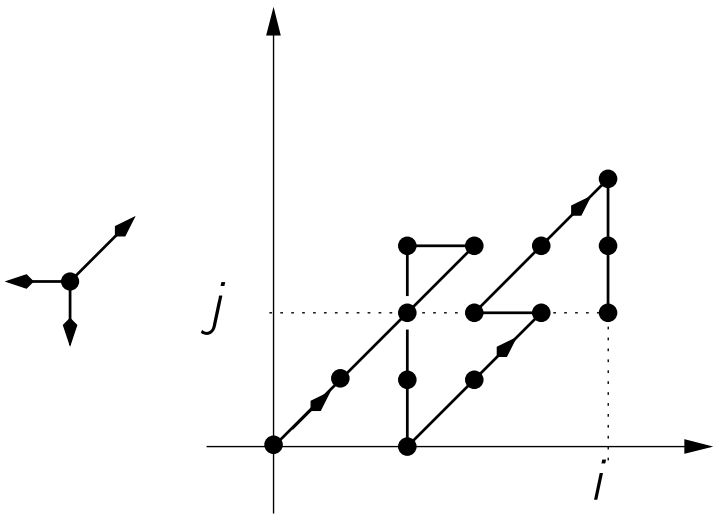


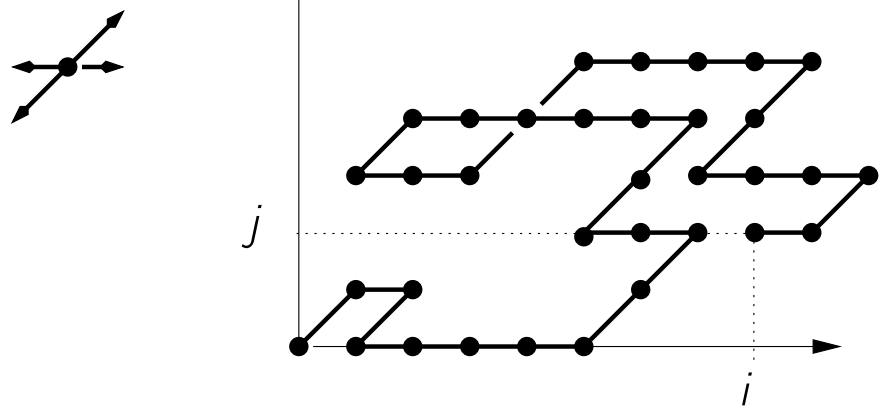


and much more!



VIII. But why these steps? Why these cones?





The whole story

Counting walks in (rational) cones

Take a starting point p_0 in $\mathbb{Z}^d$, a (finite) step set $\mathcal{S} \subset \mathbb{Z}^d$ and a cone $\mathcal{C}$.

Questions

- What is the number $c(n)$ of n-step walks starting at p_0, taking their steps in $\mathcal{S}$ and contained in $\mathcal{C}$?
- For $\boldsymbol{i} = (i_1, \ldots, i_d) \in \mathcal{C}$, what is the number $c(\boldsymbol{i}; n)$ of such walks that end at $\boldsymbol{i}$?

Generating function:

$$C(\boldsymbol{x}; t) = \sum_{\boldsymbol{i}, n} c(\boldsymbol{i}; n) x_1^{i_1} \cdots x_d^{i_d} t^n$$

A (very) basic cone: the full space

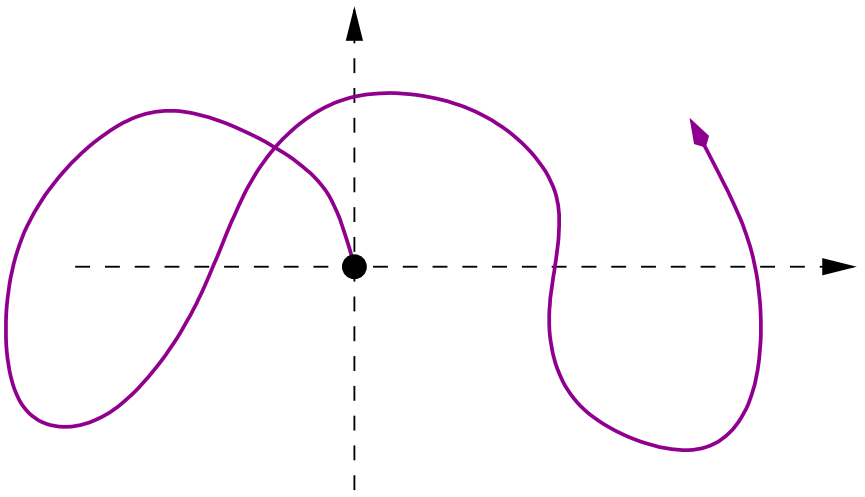
Rational series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $\mathsf{C}(\boldsymbol{x}; t)$ is rational:

$$c(n) = |\mathcal{S}|^n \quad \Leftrightarrow \quad \mathsf{C}(t) = \sum_{n \geq 0} c(n) t^n = \frac{1}{1 - |\mathcal{S}|\, t}$$

More generally:

$$\mathsf{C}(\boldsymbol{x}; t) = \frac{1}{1 - t \sum_{s \in \mathcal{S}} \boldsymbol{x}^{\boldsymbol{s}}}.$$

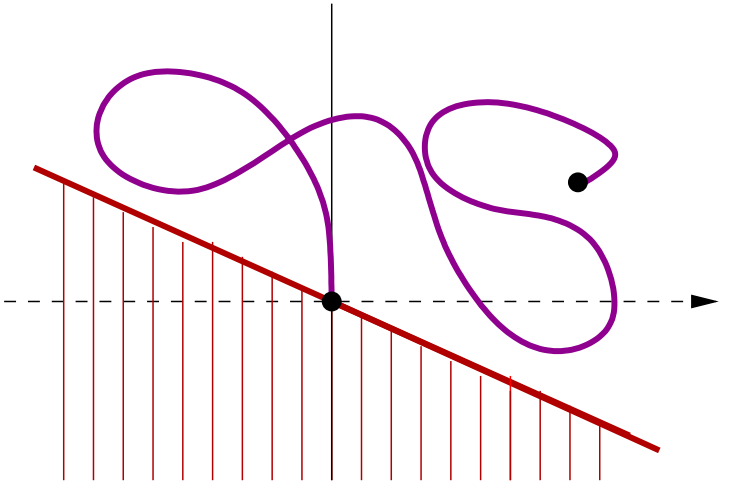


Also well-known: a (rational) half-space

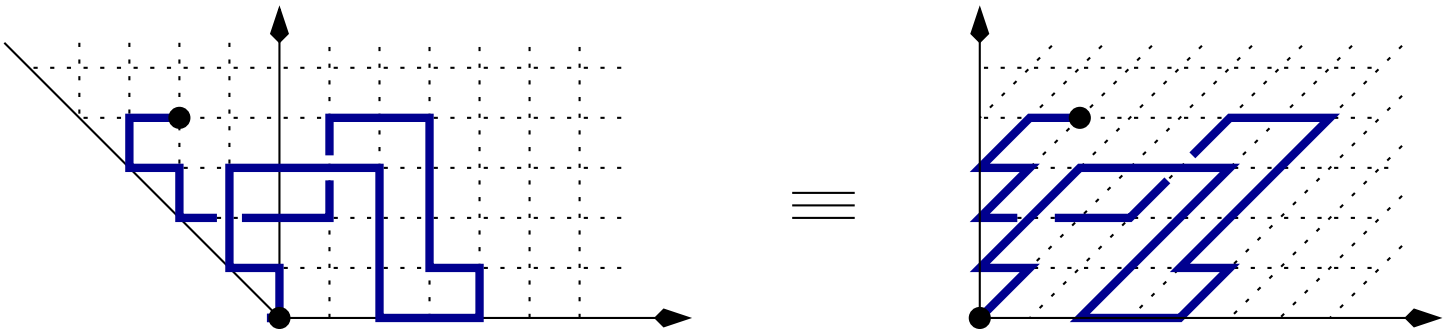
Algebraic series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and C is a rational half-space, then $C(\boldsymbol{x}; t)$ is algebraic, given by an explicit system of polynomial equations.

[Labelle, Yeh 90, Labelle 93, Duchon 00, mbm-Petkovšek 00, Banderier & Flajolet 02, ...]

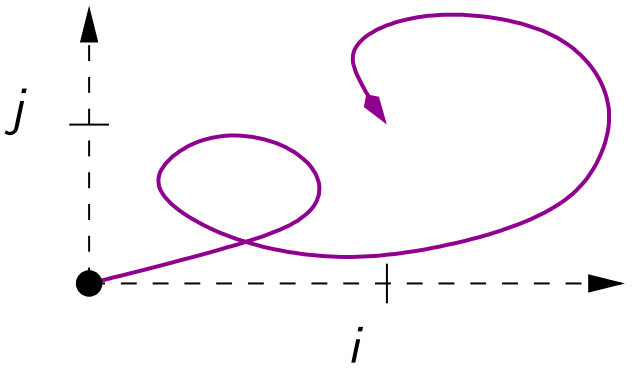


2D walks in a convex cone $\Rightarrow$ walks in the first quadrant



- The quarter plane, with $p_0 = (0,0)$, and steps $\mathcal{S} \subset \mathbb{Z}^2$:

$$Q(x,y;t) = \sum_{i,j,n \geq 0} q(i,j;n) x^i y^j t^n = ?$$



A hierarchy of formal power series

• The formal power series $\mathsf{C}(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, \mathsf{C}(t)) = 0.$$

A hierarchy of formal power series

• The formal power series $C(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, C(t)) = 0.$$

• The formal power series $C(t)$ is D-finite (holonomic) if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_k(t)C^{(k)}(t) + \cdots + P_0(t)C(t) = 0.$$

◦ Nice and effective closure properties + asymptotics of the coefficients

A hierarchy of formal power series

• The formal power series $C(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, C(t)) = 0.$$

• The formal power series $C(t)$ is D-finite (holonomic) if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_k(t)C^{(k)}(t) + \cdots + P_0(t)C(t) = 0.$$

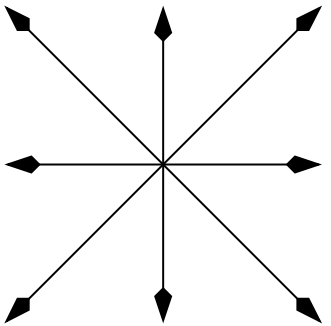
◦ Nice and effective closure properties + asymptotics of the coefficients
◦ Extension to several variables (D-finite: one DE per variable)

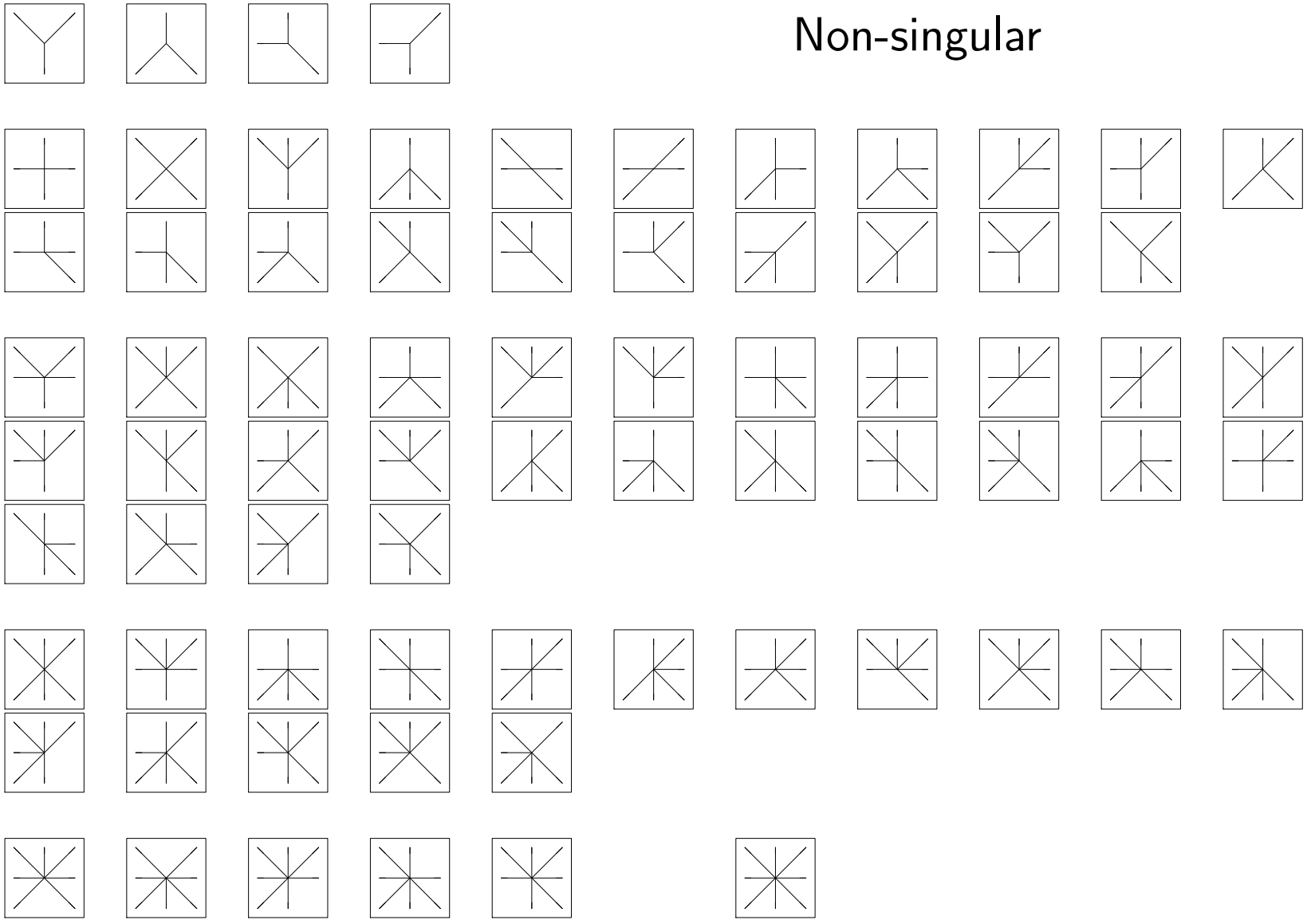
Quadrant walks with *small* steps: classification

- $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\} \Rightarrow 2^8 = 256$ step sets (or: models)

- However, some models are equivalent to a half-space problem (hence algebraic) and/or to another model (diagonal symmetry).

- One is left with 79 interesting distinct models [mbm-Mishna 09].




Non-singular
Singular

Classification of quadrant walks with small steps

Theorem (for the 79 interesting models)

The series $Q(x, y; t)$ is D-finite if and only if a certain group G associated with $\mathcal{S}$ is finite.
It is algebraic iff, in addition, the orbit sum is zero.

[mbm-Mishna 10], [Bostan-Kauers 10]	D-finite
[Kurkova-Raschel 12]	non-singular non-D-finite
[Mishna-Rechnitzer 07], [Melczer-Mishna 13]	singular non-D-finite

$\Rightarrow$ Classification of the corresponding functional equations, e.g.

$$\left(1 - t(x + \bar{x} + y + \bar{y})\right) Q(x, y; t) = 1 - t\bar{x}Q(0, y; t) - t\bar{y}Q(x, 0; t)$$

The group of the model

To a step set $\mathcal{S}$, associate the step polynomial:

$$S(x,y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j.$$

The group is generated by two birational transformations of (x,y) that leave $S(x,y)$ unchanged.

Example. For the square lattice,

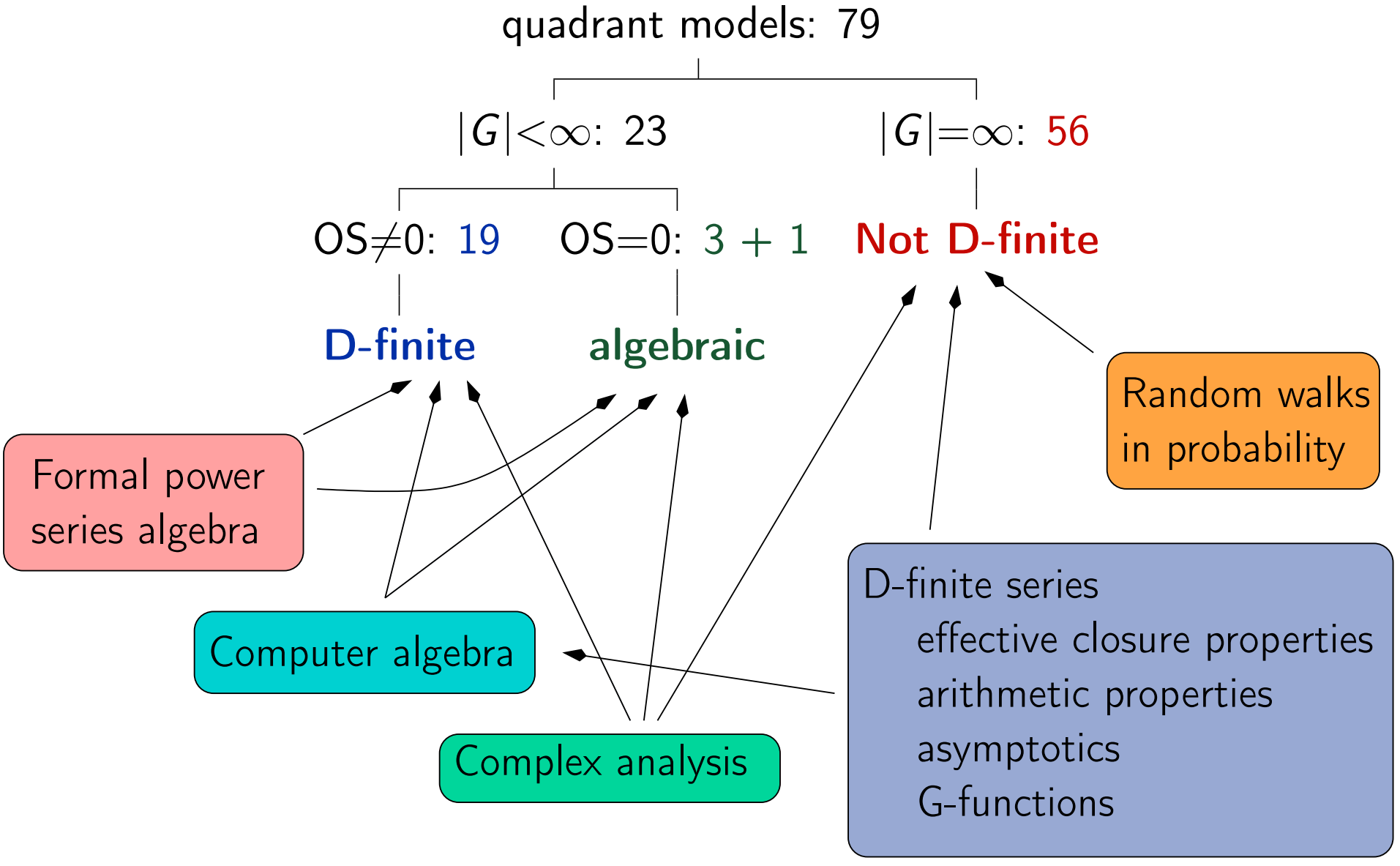
$$S(x,y) = \frac{1}{x} + x + y + \frac{1}{y} = \bar{x} + x + y + \bar{y}$$

and

$$\Phi : (x,y) \mapsto (\bar{x},y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x,\bar{y})\,.$$

These two involutions generate a group of order 4.

Classification of quadrant walks with small steps



More steps, more cones

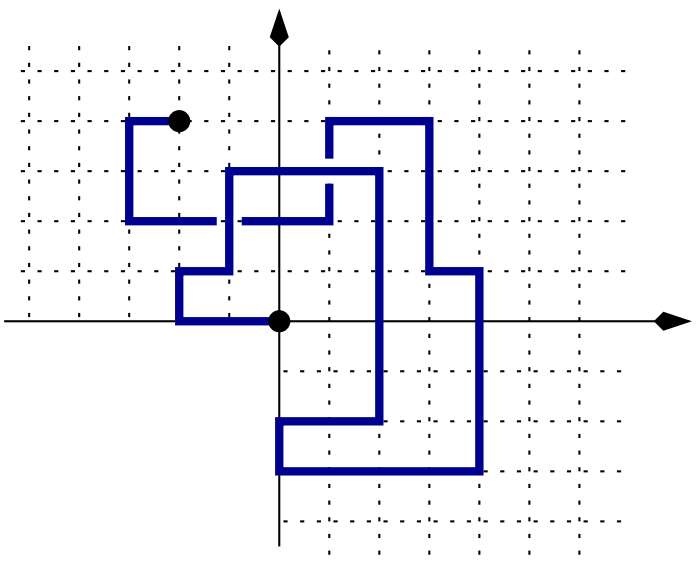
- Quadrant walks with larger steps [Fayolle, Raschel 15], [Bostan, mbm, Melczer 16?]
- 3D walks in the non-negative octant [Bostan, mbm, Kauers, Melczer 14(a), Bacher, Kauers, Yatchak 15(a)]
- Non-convex cones: 2D walks avoiding the negative quadrant [mbm 15(a)], [Mustapha 16].

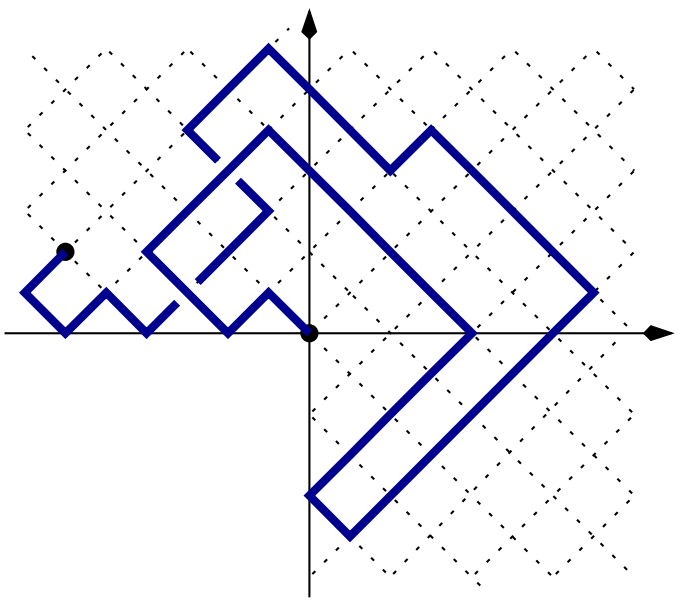
More steps, more cones

- Quadrant walks with larger steps [Fayolle, Raschel 15], [Bostan, mbm, Melczer 16?]
- 3D walks in the non-negative octant [Bostan, mbm, Kauers, Melczer 14(a), Bacher, Kauers, Yatchak 15(a)]
- Non-convex cones: 2D walks avoiding the negative quadrant [mbm 15(a)], [Mustapha 16].

More steps, more cones

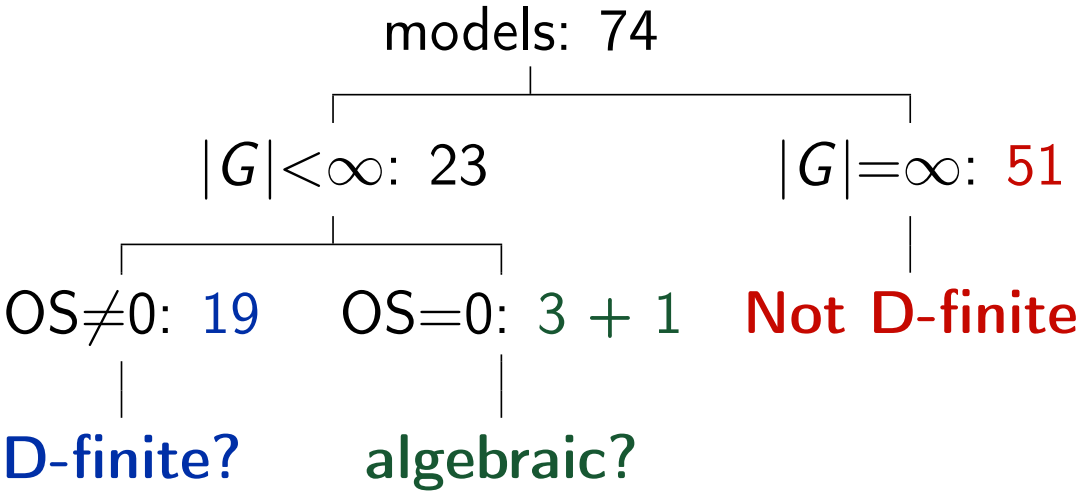
- Quadrant walks with larger steps [Fayolle, Raschel 15], [Bostan, mbm, Melczer 16?]
- 3D walks in the non-negative octant [Bostan, mbm, Kauers, Melczer 14(a), Bacher, Kauers, Yatchak 15(a)]
- Non-convex cones: 2D walks avoiding the negative quadrant [mbm 15(a)], [Mustapha 16].





More steps, more cones

- Quadrant walks with larger steps [Fayolle, Raschel 15], [Bostan, mbm, Melczer 16?]
- 3D walks in the non-negative octant [Bostan, mbm, Kauers, Melczer 14(a), Bacher, Kauers, Yatchak 15(a)]
- Non-convex cones: 2D walks avoiding the negative quadrant [mbm 15(a)], [Mustapha 16].



Final comments

- Many combinatorial classes with an algebraic GF are not "context-free" (not $\mathbb{N}$-algebraic). A better understanding is needed!
- Is there a combinatorial class with a rational, but not $\mathbb{N}$-rational, GF?
- D-finite series are nice (combinatorial interpretations [Kotek, Makowsky], [Massazza], [Pak]...)
- (Very) hard algebraic problems: those for which we do not know any recursive description.

Final comments

- Many combinatorial classes with an algebraic GF are not "context-free" (not $\mathbb{N}$-algebraic). A better understanding is needed!
- Is there a combinatorial class with a rational, but not $\mathbb{N}$-rational, GF?
- D-finite series are nice (combinatorial interpretations [Kotek, Makowsky], [Massazza], [Pak]...)
- (Very) hard algebraic problems: those for which we do not know any recursive description.

Final comments

- Many combinatorial classes with an algebraic GF are not "context-free" (not $\mathbb{N}$-algebraic). A better understanding is needed!
- Is there a combinatorial class with a rational, but not $\mathbb{N}$-rational, GF?
- D-finite series are nice (combinatorial interpretations [Kotek, Makowsky], [Massazza], [Pak]...)
- (Very) hard algebraic problems: those for which we do not know any recursive description.

Final comments

- Many combinatorial classes with an algebraic GF are not "context-free" (not $\mathbb{N}$-algebraic). A better understanding is needed!
- Is there a combinatorial class with a rational, but not $\mathbb{N}$-rational, GF?
- D-finite series are nice (combinatorial interpretations [Kotek, Makowsky], [Massazza], [Pak]...)
- (Very) hard algebraic problems: those for which we do not know any recursive description.