

RSK correspondence, jeu de taquin, and growth diagrams

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Standard Young tableaux

A *standard Young tableau* is a left-justified array of the integers $1, 2, \dots, n$, such that entries are increasing along rows and columns.

Given a standard Young tableau T , the vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, where λ_i is the length of the i -th row of T is called the *shape* of T .

Example

$n = 12$, $\lambda = (4, 3, 2, 2, 1)$.

1	3	4	8
2	6	7	
5	10		
9	12		
11			

Robinson–Schensted correspondence

The Robinson–Schensted correspondence is a bijection

$$\pi \longleftrightarrow (P, Q)$$

between *permutations* of $\{1, 2, \dots, n\}$ and pairs (P, Q) of *standard Young tableaux of the same shape*.

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Hence,

$$n! = \sum_{\lambda \vdash n} f_{\lambda}^2,$$

where f_{λ} denotes the number of standard Young tableaux of shape λ .

Robinson–Schensted correspondence

Example

$\pi = 5137426$:

$$\emptyset \longrightarrow \begin{array}{c} 5 \end{array} \longrightarrow \begin{array}{c} 1 \\ 5 \end{array} \longrightarrow \begin{array}{c} 13 \\ 5 \end{array} \longrightarrow \begin{array}{c} 137 \\ 5 \end{array}$$
$$\longrightarrow \begin{array}{c} 134 \\ 57 \end{array} \longrightarrow \begin{array}{c} 124 \\ 37 \\ 5 \end{array} \longrightarrow \begin{array}{c} 1246 \\ 37 \\ 5 \end{array}$$

Example

$\pi = 5137426$:

$$\begin{aligned}(\emptyset, \emptyset) &\longrightarrow (5, 1) \longrightarrow \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 13 & 13 \\ 5 & 2 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 137 & 134 \\ 5 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 134 & 134 \\ 57 & 25 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 124 & 134 \\ 37 & 25 \\ 5 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1246 & 1347 \\ 37 & 25 \\ 5 & 6 \end{pmatrix}\end{aligned}$$

Growth diagrams

Growth diagrams

(Fomin)

\emptyset	1	11	21	31	32	321	421
			X				
\emptyset	1	11	21	21	31	311	411
\emptyset	1	11	21	21	31	311	311
X	\emptyset	1	2	2	3	31	31
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\emptyset	\emptyset	X	1	1	1	2	2
\emptyset	\emptyset	1	1	1	1	1	1
\emptyset	X	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset

Growth diagrams

Informally:

Growth diagrams are diagrams consisting of square cells.

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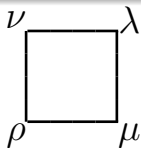
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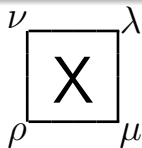
The corners of cells are labelled by partitions.

Along each row or column of a diagram, partitions are contained in each other, growing by at most one at a time. Growth by one can only happen if in the row/column to the left/below there is an X.

Growth diagrams



a. A cell without cross

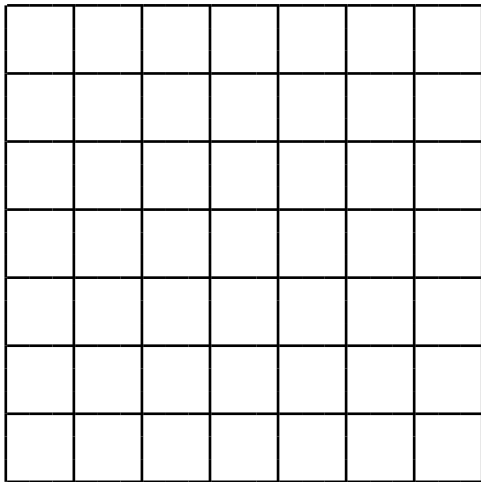


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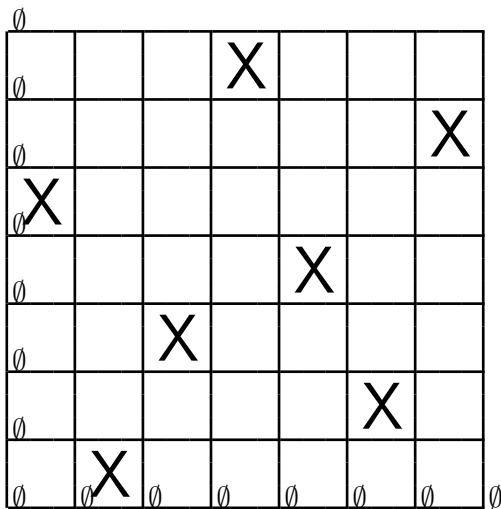
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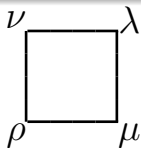
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						X
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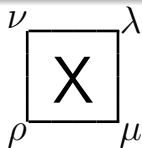
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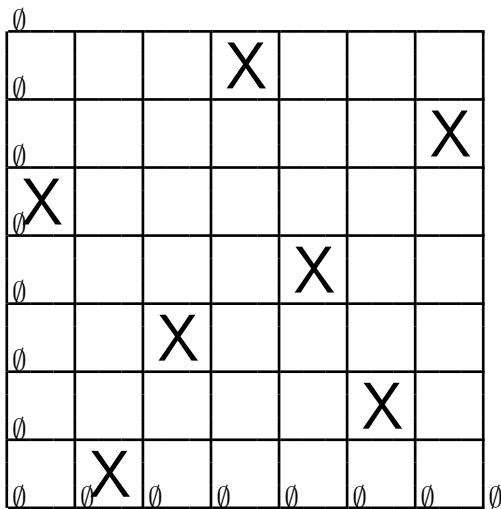


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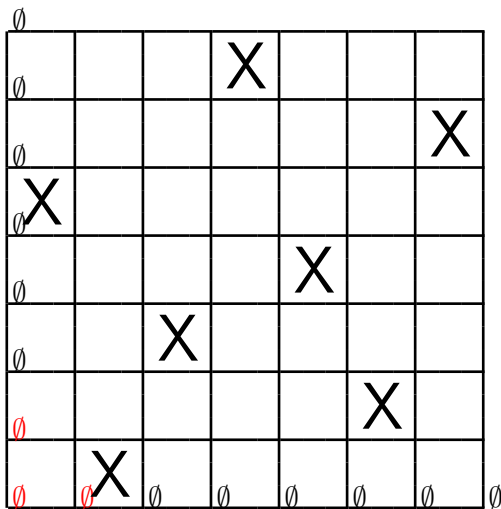
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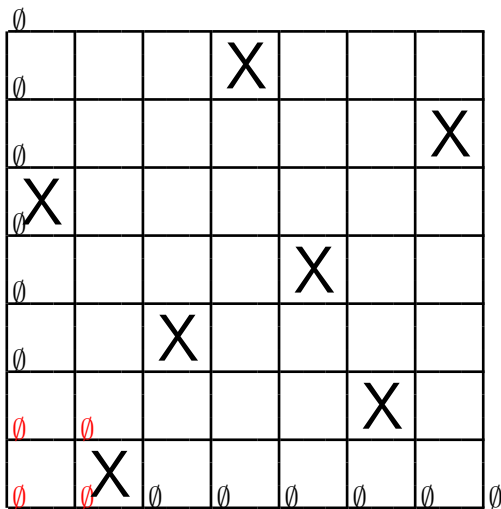
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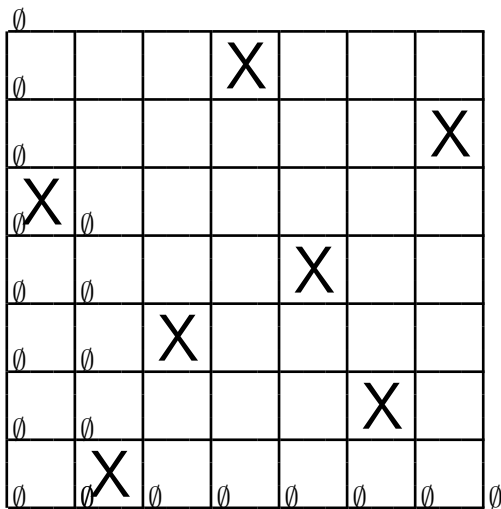
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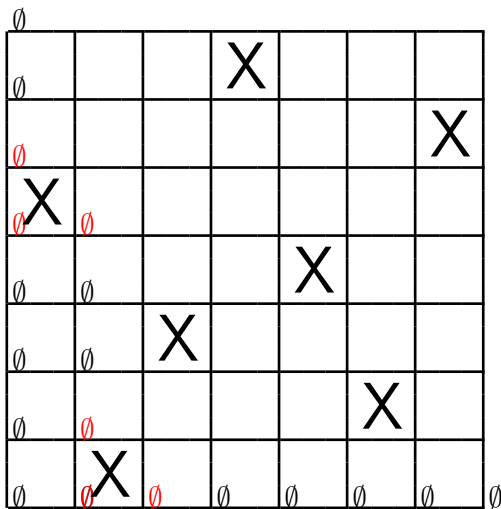
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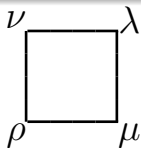


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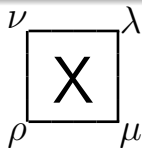
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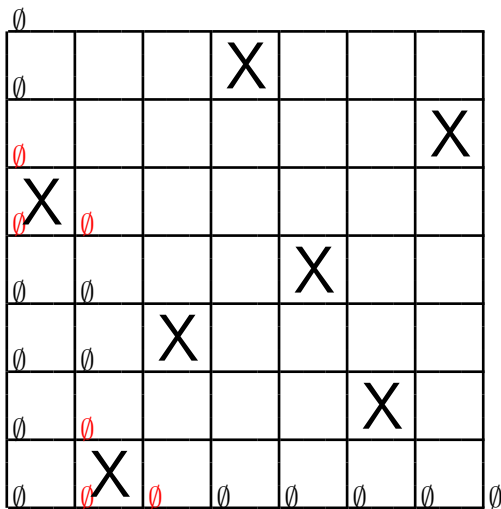


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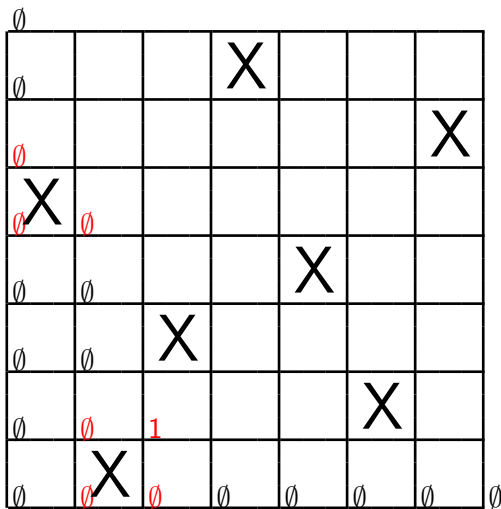
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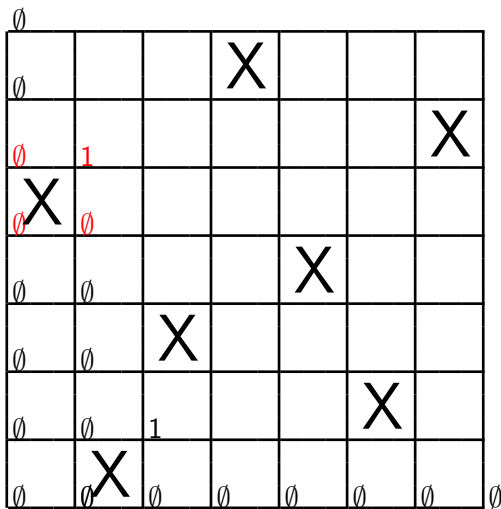
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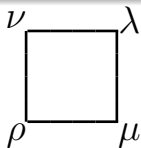


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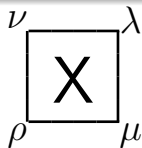
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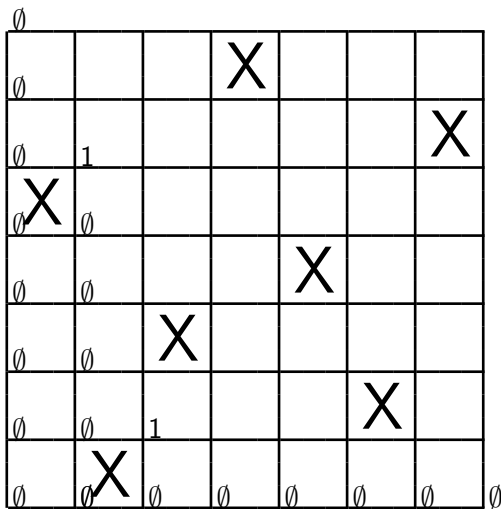


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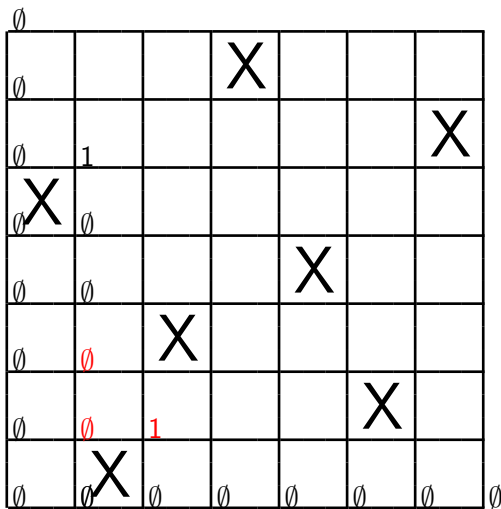
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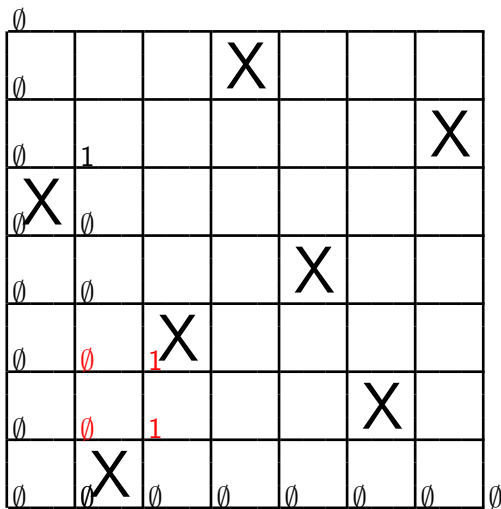
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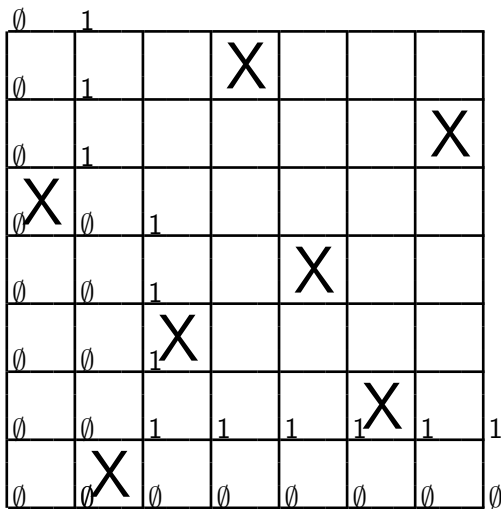
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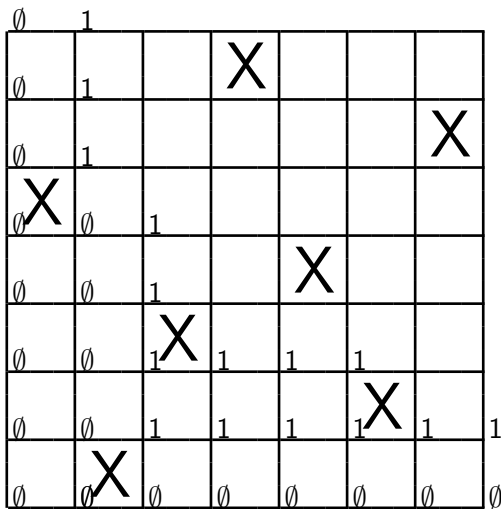
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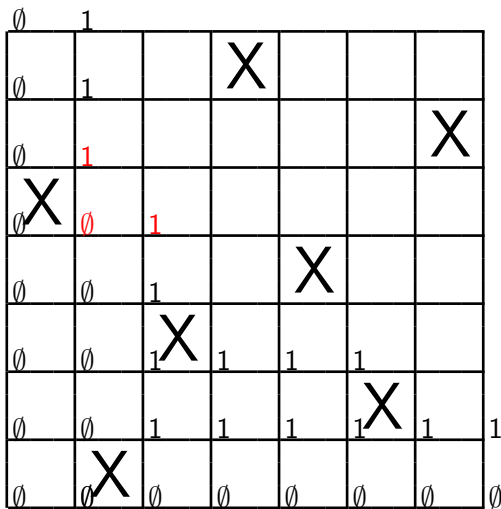
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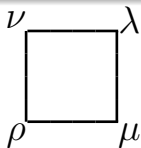


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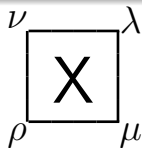
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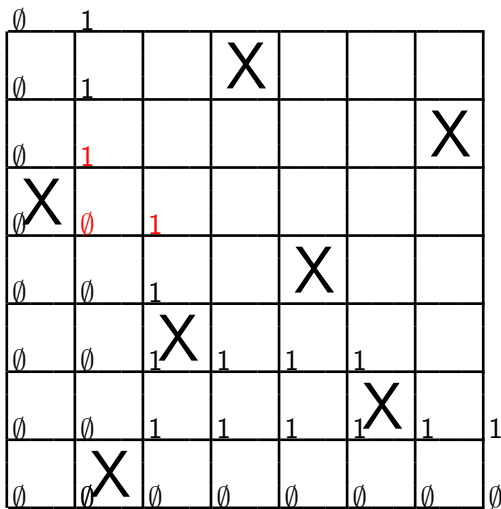


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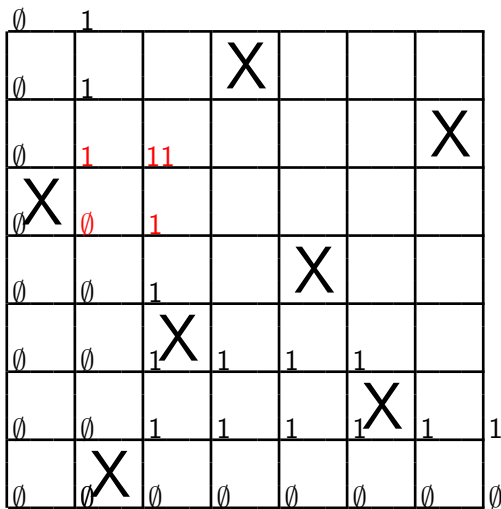
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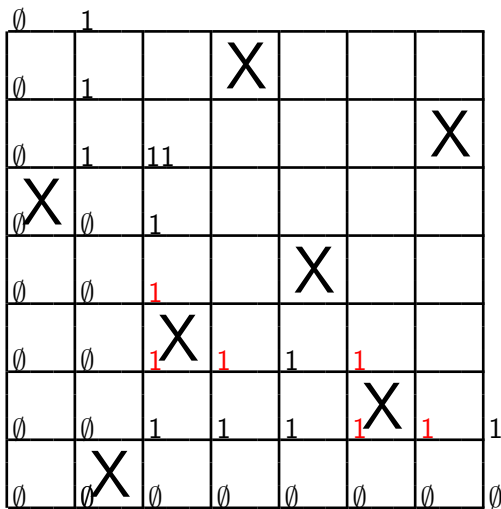
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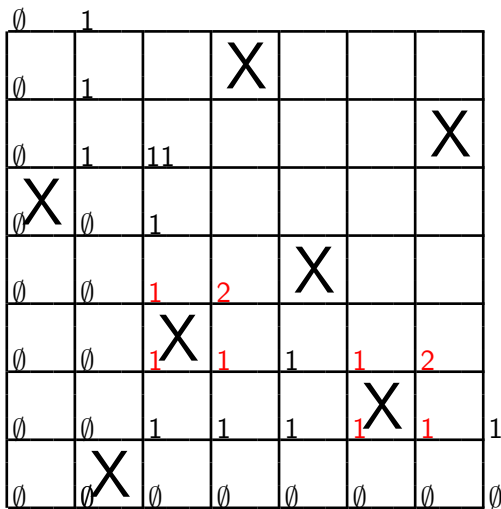
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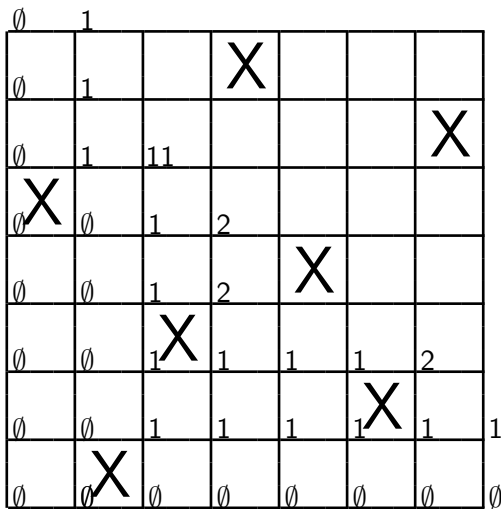
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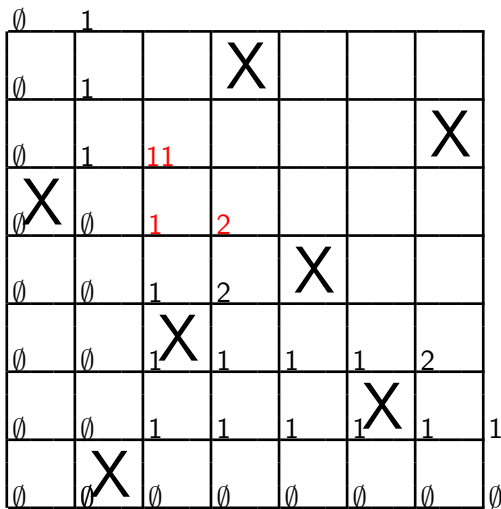
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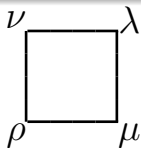


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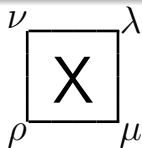
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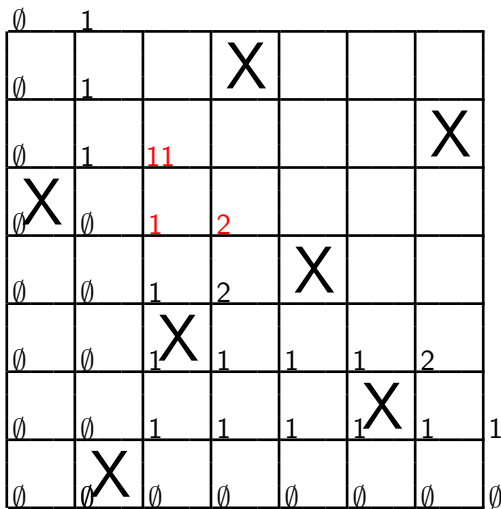


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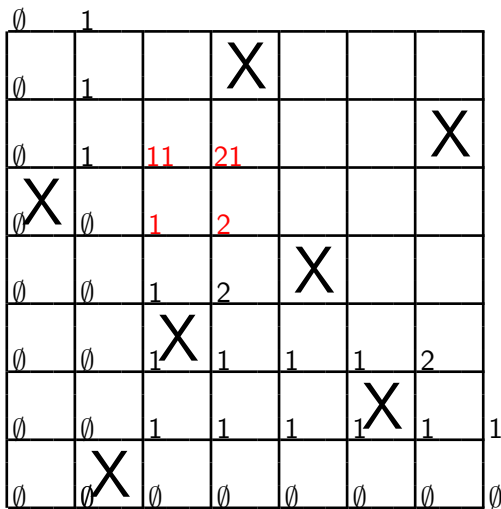
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Growth diagrams

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\emptyset	1	11	21	31	32	321	421
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X	\emptyset	1	2	2	3	31	31
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\emptyset	\emptyset	X	1	1	1	2	2
\emptyset	\emptyset	1	1	1	X	1	1
\emptyset	X	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset

Growth diagrams \longleftrightarrow Robinson–Schensted correspondence

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Along the right border of the diagram we read:

$\emptyset, 1, 2, 21, 31, 311, 411, 421.$

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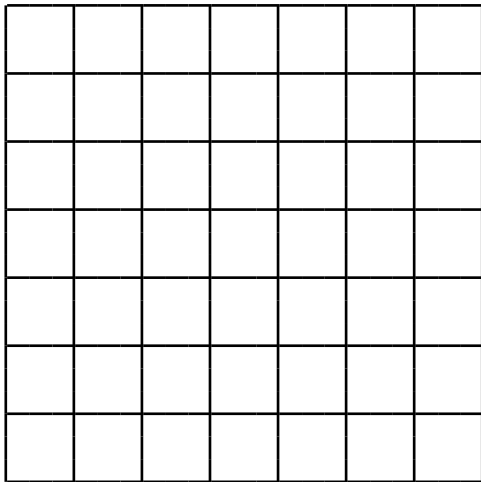
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These two sequences of shapes correspond to the pair

$$\left(\begin{array}{cc} 1246 & 1347 \\ 37 & 25 \\ 5 & 6 \end{array} \right).$$

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X						
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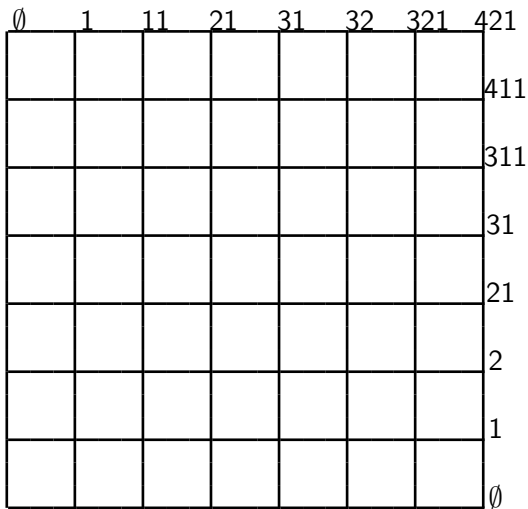
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\emptyset	X	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset

Growth diagrams — global view

Theorem (GREENE)

Given a growth diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ labelling corner c satisfies the following two properties:

- (G1) For any k , the maximal cardinality of the union of k increasing chains situated in the rectangular region to the left and below of c is equal to $\lambda_1 + \lambda_2 + \dots + \lambda_k$.*
- (G2) For any k , the maximal cardinality of the union of k decreasing chains situated in the rectangular region to the left and below of c is equal to $\lambda'_1 + \lambda'_2 + \dots + \lambda'_k$, where λ' denotes the partition conjugate to λ .*

In particular, λ_1 is the length of the longest increasing chain in the rectangular region to the left and below of c , and λ'_1 is the length of the longest decreasing chain in the same rectangular region.

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Conjecture (BURRILL)

Let n and k be given non-negative integers. The number of oscillating tableaux of length n with at most k columns, starting at \emptyset and ending at some one-column shape is the same as the number of standard Young tableaux of size n with all columns of length at most $2k$.

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(A standard Young tableau is an oscillating tableau where λ^{i+1} is always by one cell larger than λ^i .)

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Let n and k be given non-negative integers. The number of oscillating tableaux of length n with at most k columns, starting at \emptyset and ending at some one-column shape is the same as the number of standard Young tableaux of size n with all columns of length at most $2k$.

Theorem

Let n, m, k be given non-negative integers. The number of oscillating tableaux of length n with at most k columns, starting at \emptyset and ending at the one-column shape (1^m) , is equal to the number of standard Young tableaux of size n with m columns of odd length, all columns of length at most $2k$.

Example

$n = 12, k = 3, m = 2.$

1	3	4	8
2	6	7	
5	10		
9	12		
11			



$\emptyset, 1, 11, 21, 22, 21, 31, 311, 3111, 311, 31, 21, 11.$

The bijection

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	
5	10		
9	12		
11			

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	//
5	10		
9	12		
11			
/			

Fill in $/$, $//$, \dots so that all columns have even length.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	//
5	10		
9	12		
11			
/			

Fill in $/, //, \dots$ so that all columns have even length.

Slide $/, //, ///, \dots$, in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	//
5	10		
9	12		
/			
11			

Fill in $/, //, \dots$ so that all columns have even length.

Slide $/, //, III, \dots$, in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	//
5	10		
/	12		
9			
11			

Fill in $/, //, \dots$ so that all columns have even length.

Slide $/, //, ///, \dots$, in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	//
/	10		
5	12		
9			
11			

Fill in $/, //, \dots$ so that all columns have even length.

Slide $/, //, III, \dots$, in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
<i>I</i>	6	7	<i>II</i>
2	10		
5	12		
9			
11			

Fill in *I*, *II*, ... so that all columns have even length.

Slide *I*, *II*, *III*, ..., in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

<i>I</i>	3	4	8
1	6	7	<i>II</i>
2	10		
5	12		
9			
11			

Fill in *I*, *II*, ... so that all columns have even length.

Slide *I*, *II*, *III*, ..., in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

<i>I</i>	3	4	<i>II</i>
1	6	7	8
2	10		
5	12		
9			
11			

Fill in *I*, *II*, ... so that all columns have even length.

Slide *I*, *II*, *III*, ..., in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

<i>I</i>	3	<i>II</i>	4
1	6	7	8
2	10		
5	12		
9			
11			

Fill in *I, II, ...* so that all columns have even length.

Slide *I, II, III, ...*, in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

<i>I</i>	<i>II</i>	3	4
1	6	7	8
2	10		
5	12		
9			
11			

Fill in *I*, *II*, ... so that all columns have even length.

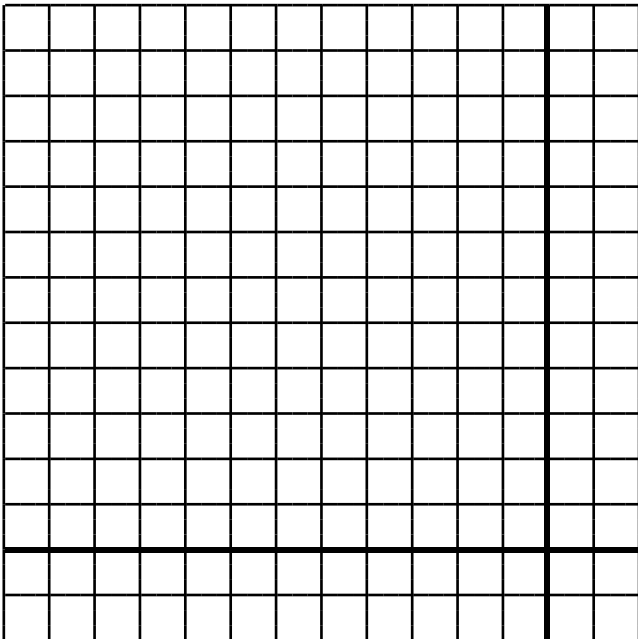
Slide *I*, *II*, *III*, ..., in this order, to the top-left of the tableau.

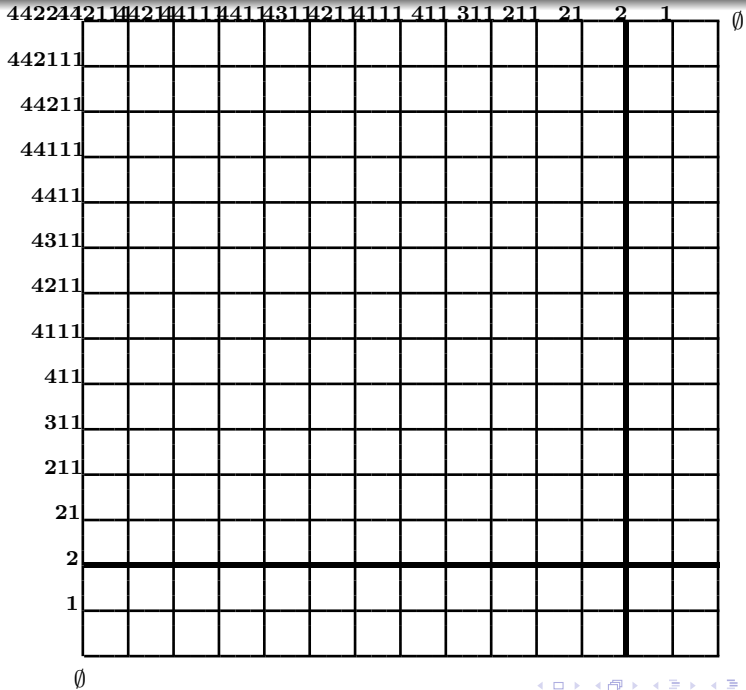
I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

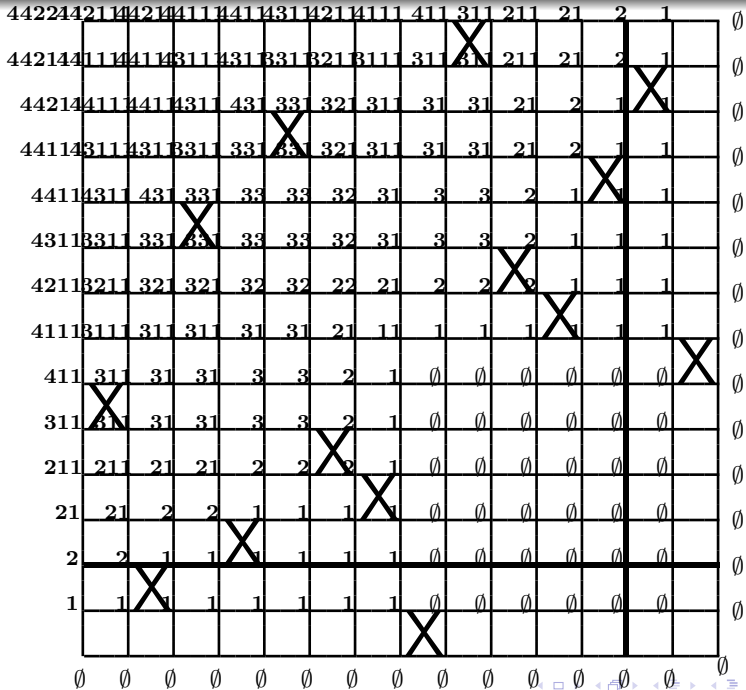
STEP 2. Put this standard Young tableaux in the alphabet

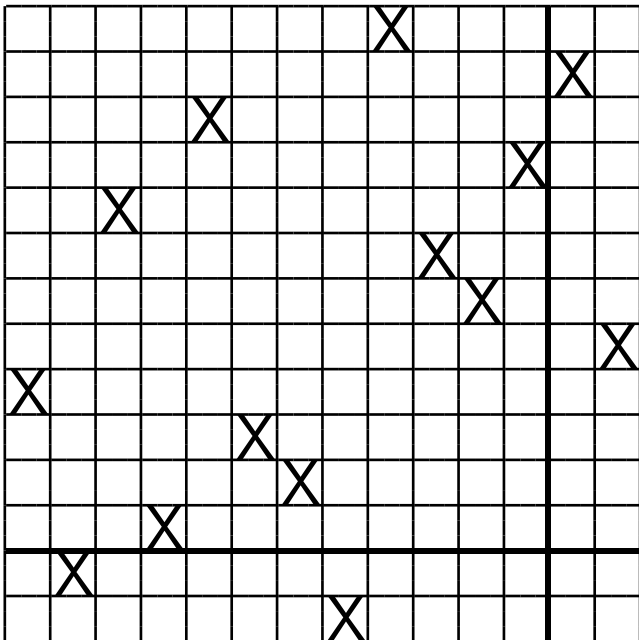
$$I, II, \dots, 1, 2, \dots$$

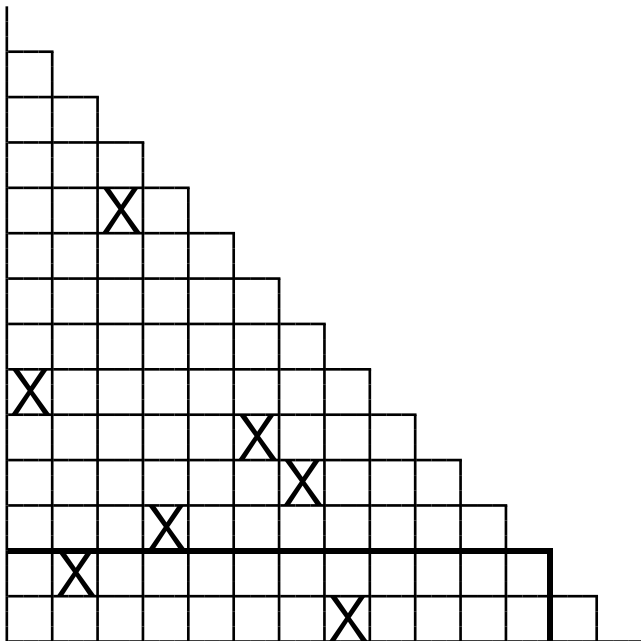
along the left and top side of a (square) growth diagram, and play the (inverse) growth diagram game in direction right/bottom.



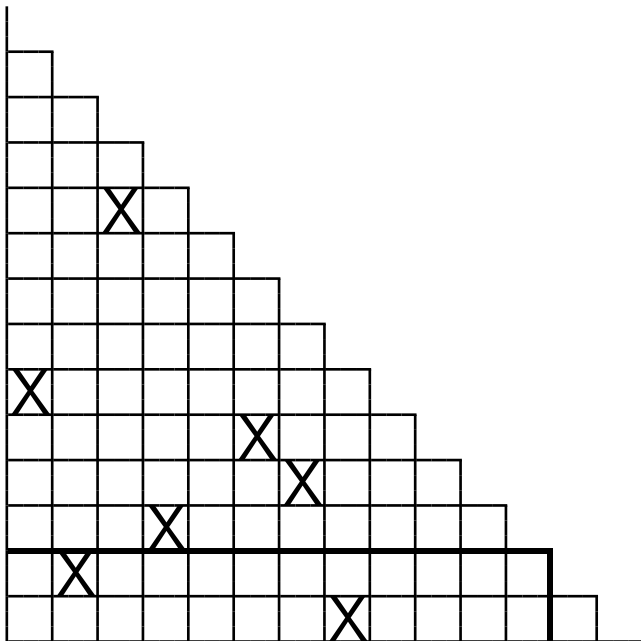


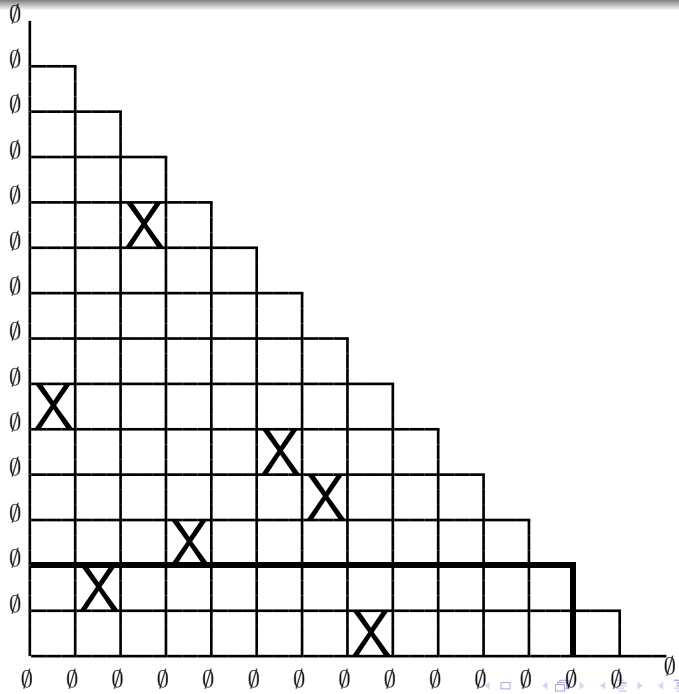


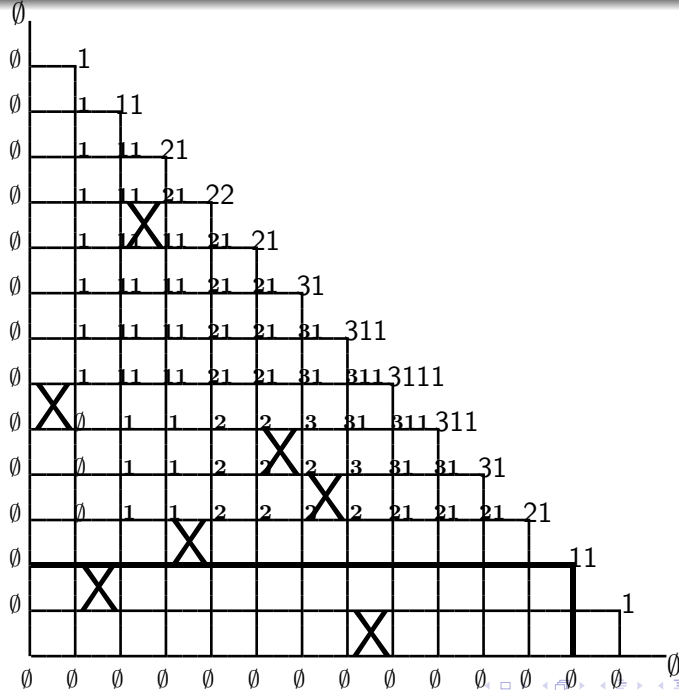


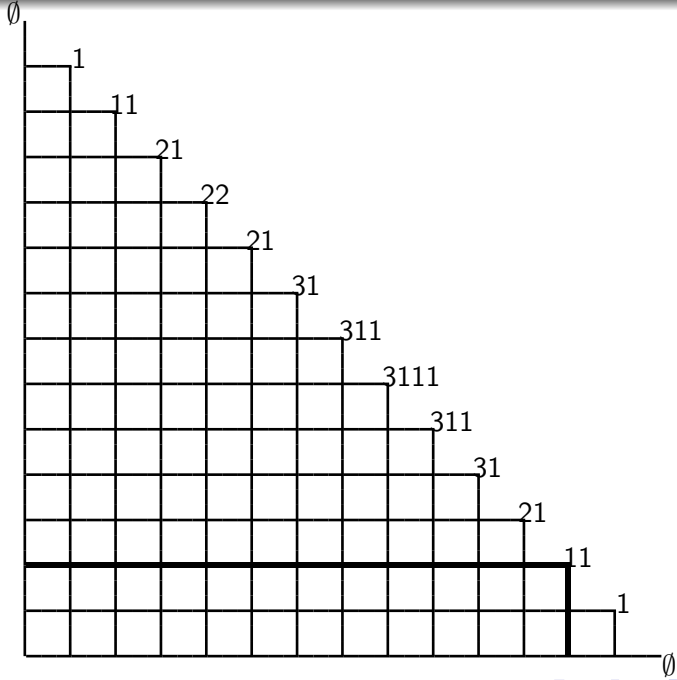


STEP 3. Place \emptyset 's along the left and bottom side of the (triangular) growth diagram, and play the (forward) growth diagram game in direction right/up.









We read along the diagonal:

\emptyset , 1, 11, 21, 22, 21, 31, 311, 3111, 311, 31, 21, 11)

We read along the diagonal:

\emptyset , 1, 11, 21, 22, 21, 31, 311, 3111, 311, 31, 21, 11)



1	3	4	8
2	6	7	
5	10		
9	12		
11			