

André polynomials and Eulerian polynomials, some recent developments

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Part I

Eugène Catalan and Désiré André

Jacques Bernoulli considered the sums $1^m + 2^m + \dots + (n-1)^m$ and their expansions as polynomials in n :

$$0^m + 1^m + 2^m + \dots + (n-1)^m = \sum_{k=0}^m \binom{m}{k} B_k \frac{n^{m+1-k}}{m+1-k}$$

for some rational number B_k now called the **Bernoulli numbers**.

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad \dots$$

They are now famous for various reasons in arithmetic and number theory, for example because of the link with Riemann's zeta function:

$$\zeta(2k) = \sum_{n \geq 1} \frac{1}{n^{2k}} = |B_{2k}| \frac{2^{2k-1}}{(2k)!} \pi^{2k}.$$

In 1877, Eugène Catalan considers in his *notes d'algèbre et d'analyse* some **integers** G_i defined by:

$$\tan x = G_1 \frac{x}{1!} + G_3 \frac{x^3}{3!} + \dots \quad \sec x = G_0 + G_2 \frac{x^2}{2!} + G_4 \frac{x^4}{4!} + \dots$$

We have $G_{2n-1} = |B_{2n}| \frac{2^{2n}(2^{2n}-1)}{2n}$.

And $G_0 = G_1 = G_2 = 1$, $G_3 = 2$, $G_4 = 5$, $G_5 = 16$, $G_6 = 61$, ...

He shows that they can be **computed** via:

$$2G_{n+1} = \binom{n}{0} G_0 G_n + \binom{n}{1} G_1 G_{n-1} + \binom{n}{2} G_2 G_{n-2} + \dots + \binom{n}{n} G_n G_0.$$

This identity implies **congruences** such as, for each odd prime p ,

$$G_{p+1} \equiv G_p \pmod{p}$$

As often when trigonometric functions are involved, there are also multiple variants, for example

$$G_{n+2} = \binom{n}{0} G_0 G_{n+1} + \binom{n}{1} G_1 G_n + \cdots + \binom{n}{n} G_{n+1} G_0$$

which implies the congruence

$$G_{p+2} \equiv G_{p+1} + G_p \pmod{p}$$

where p is an odd prime number.

The identities on G_n are equivalent to **differential equations** on the trigonometric functions: if $f = \tan + \sec$, we have $f' = \frac{1}{2}(1 + f^2)$, and $f'' = f'f$.

A few years later in 1881, Désiré André introduces **alternating permutations**, that are permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$, or $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$.

He shows: $2G_n$ is the number of alternating permutations in \mathfrak{S}_n .

For $n = 4$: 1324, 1423, 2413, 2314, 3412
(+5 others obtained by symmetry), 2×5 in total.

Développements de séc x et de tang x ,
Comptes rendus de l'Académie des sciences 88(1879), 965–967.

Désiré André: *Sur les permutations alternées,*
Journal de math. pures et appliquées, 3e série, 7(1881), 167–184.

Désiré André also makes remarks about **elliptic functions** $\lambda(x)$ and $\mu(x)$ satisfying

$$\begin{aligned}\lambda'^2 &= 1 - (1 + k^2)\lambda^2 + k^2\lambda^4, \\ \mu'^2 &= 1 - k^2 + (2k^2 - 1)\mu - k^2\mu^4.\end{aligned}$$

In the expansions

$$\lambda(x) = x - P_1 \frac{x^3}{3!} + P_2 \frac{x^5}{5!} - \dots, \quad \mu(x) = 1 - Q_1 \frac{x^2}{2!} + Q_2 \frac{x^4}{4!} - \dots,$$

the coefficients P_i and Q_i are polynomials in k^2 .

Their values at $k = 1$ are respectively G_{2i+1} and G_{2i} , and he implicitly suggests the problem of finding the related combinatorial objects.

Part II

Dominique Foata and Marcel-Paul Schützenberger

In the early 70's, Dominique Foata and Marcel-Paul Schützenberger published two monographs:

Théorie géométrique des polynômes eulériens, Berlin, Springer-Verlag, 1970 (Lecture Notes in Math. 138), 94 pages.

Nombres d'Euler et permutations alternantes, Technical Report, University of Florida, 1971, 71 pages.

They introduce a variety of new objects such as **André permutations**, **André trees**, **André polynomials**, **André complexes**...

They consider the differential equation

$$f'' = t \exp(f), \quad f(0) = 0, \quad f'(0) = s$$

where s and t are some parameters. The coefficients $D_n(s, t)$ in the expansion $f(u) = \sum_{n \geq 0} D_n \frac{u^n}{n!}$ are polynomials in s and t , called **André polynomials**.

$$D_0 = 0, \quad D_1 = s, \quad D_2 = t, \quad D_3 = st, \quad D_4 = s^2t + t^2,$$

$$D_5 = s^3t + 4st^2, \quad D_6 = s^4t + 11s^2t^2 + 4t^3, \quad \dots$$

Concretely, they can be computed by $D_{n+1} = st\partial_t D_n + t\partial_s D_n$ for $n \geq 1$. And $D_n(1, 1) = G_{n-1}$.

Why such a definition ? The goal is to show that properties of the numbers G_n can be generalized at the level of polynomials.

“This memoir is intended in our mind to prepare an analysis of arithmetical properties of the numbers G_n .” [FS71].

They are related with **Eulerian polynomials** $A_n(x)$ satisfying

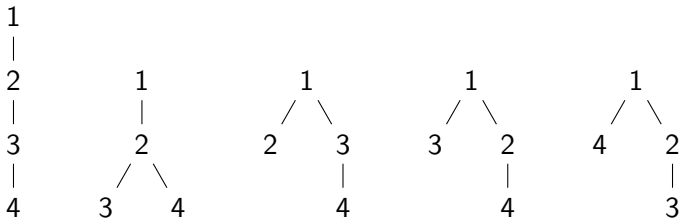
$$\sum_{k \geq 0} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}$$

by

$$A_{n-1}(x) = \frac{(1+x)^n}{2x} D_n\left(1, \frac{2x}{(1+x)^2}\right).$$

An **André tree** on n vertices is such that:

- ▶ vertices are labelled with the integers $1, \dots, n$ increasingly from the root to the leaves,
- ▶ each internal vertex has either one descendant, or two unordered descendants.



are the 5 André trees on 4 vertices.

Their original definition is as follows:

Let x_1, x_2, \dots, x_n be the increasing sequence of elements in a finite set X , totally ordered and of cardinality n .

A map f from X to X is a binary decreasing arborescence if

- (1) $xf < x$ for all $x \in X \setminus x_1$;
- (2) $x_1 f = x_1$;
- (3) $\text{Card}[xf^{-1} \setminus \{x\}] \geq 2$ for all $x \in X$.

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Theorem

$D_n(s, t) = \sum_T s^{\alpha(T)} t^{\beta(T)}$ where we sum over André trees on n vertices, $\alpha(T)$ is the numbers vertices with one descendant, $\beta(T)$ the number of leaves.

They also define a structure of **André complex**, a formalisation of André trees.

- ▶ Let \mathcal{Y} be a combinatorial class, i.e., a sequence of sets $(Y_i)_{i \geq 0}$. Assume $Y_0 = \{e\}$.
- ▶ Let $Y_n^{(2)}$ be the set of pairs $\{(f_1, X_1), (f_2, X_2)\}$ where
 - ▶ X_1 and X_2 form a partition of $\{2, \dots, n\}$,
 - ▶ $f_j \in Y_{n_j}$ with $n_j = \text{Card}X_j$ for $j = 1, 2$.
- ▶ Then (Y, ϕ) is an André complex if ϕ is a collection of bijections $\phi_n : Y_n \rightarrow Y_n^{(2)}$.

Two combinatorial classes bearing a structure of André complex are automatically in bijection. This idea is used to build a bijection between André trees and **André permutations**.

This structure might seem quite natural if you already know about some theory that appeared in the 80's :

- ▶ **species theory** [Joyal, Bergeron, Labelle, Leroux].
- ▶ **symbolic methods** [Flajolet].
- ▶ combinatorial theory of **differential equations** [Leroux, Viennot].

The premise available at the time was Foata's **composé partitionnel**.

The idea to encode the differential equation $f' = \frac{1}{2}(1 + f^2)$.

Part III

Modern times

Various combinatorial aspects of **André combinatorics** were studied by Foata and Strehl, Foata and Han, Hetyei, Purtill, Simion and Sundaram (inventor of **simsum permutations**), Disanto... For example André polynomials appears:

- ▶ in geometry of polytopes, as the **cd-index** of a simplex [Fine, Bayer, Purtill].
- ▶ in representation theory, as the character of Stanley's representation of \mathfrak{S}_{n+1} on maximal chains of set partitions of $\{1, \dots, n+1\}$
- ▶ in symplectic reflection algebras in positive characteristic [Northon].

Besides André polynomials, other polynomial generalization of Euler numbers are provided by Jacobian elliptic functions, $\operatorname{sn}(z)$, $\operatorname{cn}(z)$, $\operatorname{dn}(z)$, defined by

$$\begin{aligned}\operatorname{sn}'(z) &= \operatorname{cn}(z) \operatorname{dn}(z), & \operatorname{sn}(0) &= 0, \\ \operatorname{cn}'(z) &= -\operatorname{sn}(z) \operatorname{dn}(z), & \operatorname{cn}(0) &= 1, \\ \operatorname{dn}'(z) &= -k^2 \operatorname{sn}(z) \operatorname{cn}(z), & \operatorname{dn}(0) &= 1.\end{aligned}$$

Combinatorial interpretations for the Taylor coefficients were studied by Viennot, Dumont, Flajolet, under the suggestion of Marcel-Paul Schützenberger.

Maybe in MPS's mind, André polynomials and elliptics functions are in the same picture.

For example, the well known equation for the Weierstrass function:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

implies

$$\wp''(z) = 6\wp(z)^2 - \frac{g_2}{2} \quad (\text{Painlevé I equation})$$

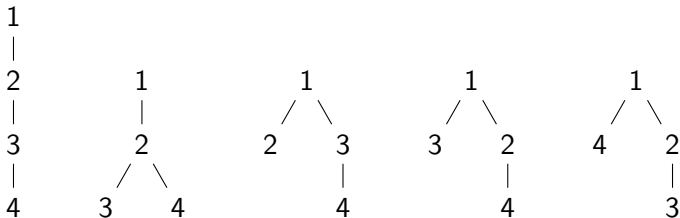
which is somewhat similar to $D''(u) = t \exp(D(u))$.

(cf. Kuba, Panholzer: Combinatorial families of multilabelled increasing trees and hook-length formulas)

In the remaining time, I will present another feature of André trees:
they give a nice **hook formula**.

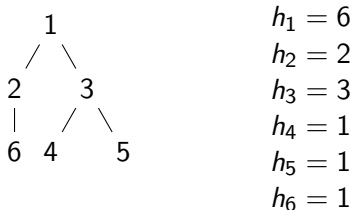
Recall that an *André tree* on n vertices is such that:

- ▶ vertices are labelled with the integers $1, \dots, n$ increasingly from the root to the leaves,
- ▶ each internal vertex has either one descendant, or two unordered descendants.



are the 5 André trees on 4 vertices.

The *hook length* h_i is the number of vertices below the one with label i .



The *weight* of an André tree T is

$$\text{wt}(T) = \prod_{\substack{1 \leq i \leq n \\ h_i(T) > 1}} \left(2 + X_i(h_i(T) - 1) \right).$$

In the example, $\text{wt}(T) = (2 + 5X_1)(2 + X_2)(2 + 2X_3)$.

Theorem (Biane and J.-V.)

$$\sum \text{wt}(T) = \prod_{i=1}^{n-1} ((n-i)X_i + i + 1)$$

where we sum over André trees on n vertices.

In general, hook formulas for trees are “easy” to prove by induction. Here, because of the multivariate weights depending on the labels, induction becomes much more difficult.

Another multivariate hook length formula for labelled trees was also obtained by Féray and Goulden. Both hook formulas reduce to Postnikov’s hook length formula for n^{n-2} in terms of Catalan trees.

- ▶ If $X_i = 0$ for all i , the weight is $2^{\# \text{ of internal vert.}}$ and the right hand side is $n!$. This weighted enumeration is a consequence of results by Foata and Strehl.
- ▶ If $X_i = x$ for all i , this formula reduces to a hook formula by Postnikov-Lascoux-Du-Liu.

What about the proof ?

We need a combinatorial set (maximal chains of noncrossing partitions), an equivalence relations (relabelling), a weight. Then:

- ▶ the total weight generating function is $\prod_{i=1}^{n-1} ((n-i)X_i + i + 1)$,
- ▶ equivalence classes are in bijection with André trees (Stanley's bijection),
- ▶ the weight generating function of the equivalence class sent to T is $\text{wt}(T)$.

Définition

A set partition of $\{1, \dots, n\}$ is noncrossing if there is no $i < j < k < l$ such that i, k are in one block, and j, l in another block.

Example

126|34|5|79A|8 is noncrossing, but 13|24 is not.



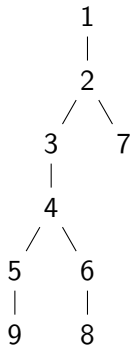
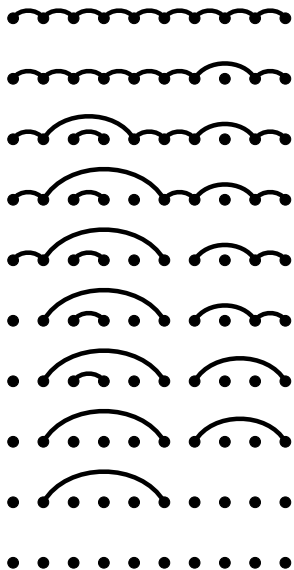
These objects are partially ordered by **refinement**.

Kreweras showed that the number of maximal chains (from $123 \dots n$ down to $1|2|3| \dots |n$) is n^{n-2} .

In general, the $(i + 1)$ th element of the chain is obtained by splitting a block of the i th element. We can distinguish two cases: either the two new blocks are *nested* or not:



In the first case, the chain gets a weight X_j . The total weight is a square free monomial in X_1, X_2, \dots



Stanley's bijection.

- ▶ On each block appearing in some element of the chain, we can perform cyclic rotations on this block to get other equivalent maximal chains. This gives the factor associated to a vertex in the corresponding André tree.
- ▶ It remains to show that the total weight generating is as expected. One possibility is to consider a slightly more general enumeration problem: minimal factorisations of a cycle in the symmetric group, which are counted by n^{k-1} (where k is the number of factors). The multivariate analog exists and can be proved by induction on the number of factors.

Thanks for your attention