

Combinatorial Problems Connected to Macdonald Polynomials

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Permutation Statistics and q -Analogues

In combinatorics a *statistic* on a finite set S is a mapping from $S \rightarrow \mathbb{N}$ given by an explicit combinatorial rule.

Ex. Given $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$, define

$$\text{inv}\pi = |\{(i, j) : i < j \text{ and } \pi_i > \pi_j\}|$$

and

$$\text{maj}\pi = \sum_{\pi_i > \pi_{i+1}} i.$$

If $\pi = 31542$,

$$\text{inv}\pi = 2 + 2 + 1 = 5$$

and

$$\text{maj}\pi = 1 + 3 + 4 = 8.$$

Let

$$\begin{aligned}(n)_q &= (1 - q^n)/(1 - q) \\ &= 1 + q + \dots + q^{n-1}\end{aligned}$$

and

$$\begin{aligned}(n!)_q &= \prod_{i=1}^n (i)_q \\ &= (1 + q)(1 + q + q^2) \cdots (1 + q + \dots + q^{n-1})\end{aligned}$$

be the q -analogues of n and $n!$. Then

$$\sum_{\pi \in S_n} q^{\text{inv}\pi} = (n!)_q = \sum_{\pi \in S_n} q^{\text{maj}\pi}.$$

Symmetric Functions

A *symmetric function* is a polynomial $f(x_1, x_2, \dots, x_n)$ which satisfies

$$f(x_{\pi_1}, \dots, x_{\pi_n}) = f(x_1, \dots, x_n),$$

i.e. $\pi f = f$, for all $\pi \in S_n$.

Examples

- The monomial symmetric functions $m_\lambda(X)$

$$m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2.$$

- The elementary symmetric functions $e_k(X)$

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3.$$

- The power-sums $p_k(X) = \sum_i x_i^k$.

- The Schur functions $s_\lambda(X)$, which are important in the representation theory of the symmetric group and GLN:

$$s_\lambda(X) = \sum_{\beta \vdash n} K_{\lambda, \beta} m_\beta(X)$$

where $K_{\lambda, \beta}$ equals the number of ways of filling the Ferrers shape of λ with elements of the multiset $\{1^{\beta_1} 2^{\beta_2} \dots\}$, weakly increasing across rows and strictly increasing down columns. For example

$$K_{(4,2), (2,2,1,1)} = 4$$

2	3			2	2		
1	1	2	4	1	1	3	4

2	4			3	4		
1	1	2	3	1	1	2	2

Plethysm: If $F(X)$ is a symmetric function, then $F[(w-1)X]$ is defined by expressing $F(X)$ as a polynomial in the $p_k(X) = \sum_i x_i^k$'s and then replacing each $p_k(X)$ by $(w^k - 1)p_k(X)$.

Theorem (Macdonald 1988)

There exists a unique basis $\{\tilde{H}_\mu(X; q, t) : \mu \vdash n\}$ for the ring of symmetric functions of total degree n characterized by the following:

- (i) $\tilde{H}_\mu[X(q-1); q, t] \in \mathbb{Q}(q, t)\{m_\lambda : \lambda \leq \mu'\}$
- (ii) $\tilde{H}_\mu[X(t-1); q, t] \in \mathbb{Q}(q, t)\{m_\lambda : \lambda \leq \mu\}$
- (iii) $\tilde{H}_\mu(1; q, t) = 1$.

where we use the “dominance” partial order on partitions;

$$\lambda \leq \mu \iff \lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i \quad 1 \leq i \leq n$$

The $n!$ Theorem

(2,0)	
(1,0)	(1,1)
(0,0)	(0,1)

$\mu = (2,2,1)$

$$\Delta(\mu) = \begin{vmatrix} 1 & y_1 & x_1 & x_1 y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2 y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3 y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4 y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5 y_5 & x_5^2 \end{vmatrix}$$

For $\mu \vdash n$ let $V(\mu)$ denote the linear span over \mathbb{Q} of all partial derivatives of all orders of $\Delta(\mu)$. $V(\mu)$ decomposes as a direct sum of its bihomogeneous subspaces $V^{i,j}(\mu)$ of degree i in the x -variables and j in the y -variables. There is an S_n -action on $V^{i,j}(\mu)$ given by

$$\pi f = f(x_{\pi_1}, \dots, x_{\pi_n}, y_{\pi_1}, \dots, y_{\pi_n})$$

called the *diagonal action*.

The *Frobenius Series* is the symmetric function

$$\sum_{\lambda \vdash n} s_\lambda(X) \sum_{i,j \geq 0} q^i t^j m_{ij},$$

where m_{ij} is the multiplicity of the irreducible S_n -character χ^λ in the module $V^{i,j}(\mu)$ induced by the diagonal action.

Theorem (Haiman; conjectured by Garsia and Haiman in the early 1990's).

The Frobenius Series of $V(\mu)$ is given by the Macdonald polynomial $\tilde{H}_\mu(X; q, t)$. In particular, the dimension of $V(\mu)$ is $|\mu|!$.

Define the $\tilde{K}_{\lambda,\mu}(q, t)$ via

$$\tilde{H}_\mu(X; q, t) = \sum_{\lambda} s_{\lambda}(X) \tilde{K}_{\lambda,\mu}(q, t).$$

As a corollary Haiman proves a famous conjecture of Macdonald, that

$$\tilde{K}_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t].$$

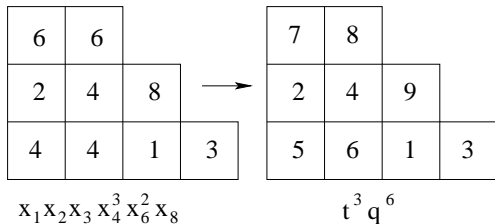
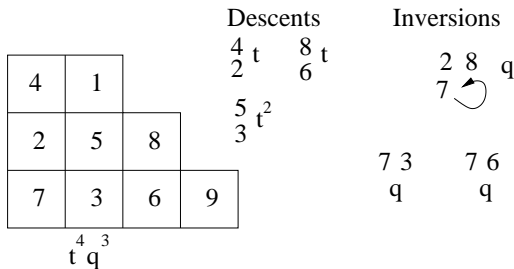
(All that can be inferred from their definition is that $\tilde{K}_{\lambda,\mu}(q, t) \in \mathbb{Q}(q, t)$. Just to prove that $\tilde{K}_{\lambda,\mu}(q, t) \in \mathbb{Z}[q, t]$ was an unsolved problem for 7 years, solved by several groups of people in 1996 – 1998.)

A related question of Macdonald is still open. He proved that $\tilde{K}_{\lambda,\mu}(1, 1) = f^\lambda$, the number of SYT of shape λ , and posed the problem of finding statistics on SYT such that

$$\tilde{K}_{\lambda,\mu}(q, t) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{qstat}(T,\mu)} t^{\text{tstat}(T,\mu)}.$$

This question has only been answered for various special shapes. In particular, Jonah Blasiak has recently proved a formula for three-column shapes conjectured by H..

The Monomial Expansion of $\tilde{H}_\mu(X; q, t)$



Reading Word: 6 6 2 4 8 4 4 1 3

Definition

Let $\text{Inv}(\sigma, \mu)$ denote the set of inversion triples, and $\text{Des}(\sigma, \mu)$ the set of descents. Set

$$\begin{aligned}\text{inv}(\sigma, \mu) &= |\text{Inv}(\sigma, \mu)|, \\ \text{maj}(\sigma, \mu) &= \sum_{w \in \text{Des}(\sigma, \mu)} 1 + \text{leg}(w)\end{aligned}$$

Remark: $\text{maj}(\sigma, 1^n) = \text{maj}(\sigma)$ and $\text{inv}(\sigma, n) = \text{inv}(\sigma)$.

Theorem H.-Haiman-Loehr (HHL) 2005

Let $X = (x_1, x_2, \dots, x_n)$. Then for any partition μ ,

$$\tilde{H}_\mu(X; q, t) = \sum_{\sigma: \mu \rightarrow \mathbb{Z}^+, \sigma_j \leq n} q^{\text{inv}(\sigma, \mu)} t^{\text{maj}(\sigma, \mu)} X^\sigma,$$

where $X^\sigma = \prod_i x_{\sigma_i}$.

Since the dimension of the irreducible S_n character χ^λ equals f^λ ,

$$\text{Hilb}(V(\mu)) = \sum_{\lambda} f^\lambda \tilde{K}_{\lambda, \mu}(q, t).$$

On the other hand,

$$\begin{aligned} \tilde{H}_\mu(X; q, t)|_{m_1^n} &= \sum_{\lambda} s_\lambda(X)|_{m_1^n} \tilde{K}_{\lambda, \mu}(q, t) \\ &= \sum_{\lambda} f^\lambda \tilde{K}_{\lambda, \mu}(q, t). \end{aligned}$$

Corollary

The Hilbert Series of $V(\mu)$ is given by

$$\sum_{\sigma \in S_n} t^{\text{maj}(\sigma, \mu)} q^{\text{inv}(\sigma, \mu)}.$$

It is an open question as to whether or not one can refine this formula and obtain an explicit basis for $V(\mu)$.

Theorem (Lascoux - Schützenberger 1978)

$$\tilde{H}_\mu(X; 0, t) = \sum_{\lambda} s_\lambda(X) \sum_{T \in \text{SSYT}(\lambda, \mu)} t^{\text{cocharge}(T)}.$$

for an intricate statistic cocharge on SSYT.

Proof: Difficult, using recurrences.

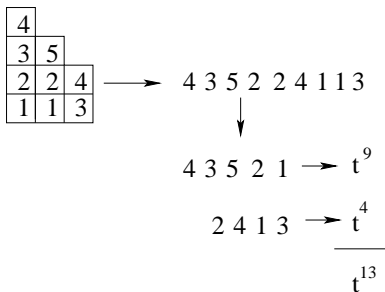
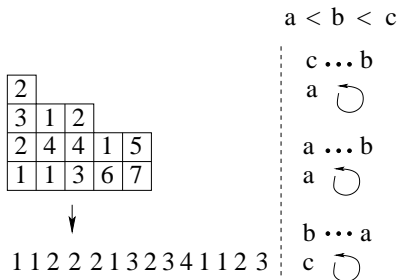


Figure: The cocharge statistic



New Proof:

$$\tilde{H}_\mu(X; 0, t) = \sum_{\sigma: \text{inv}(\sigma, \mu) = 0} t^{\text{maj}(\sigma, \mu)} x^\sigma.$$

Well known properties of the RSK alg. yield the Schur expansion:

$$\sum_{\lambda} \left(\sum_{P \in \text{SSYT}(\lambda, \mu)} t^{\text{cocharge}(P)} \right) \left(\sum_{Q \in \text{SSYT}(\lambda)} x^Q \right)$$

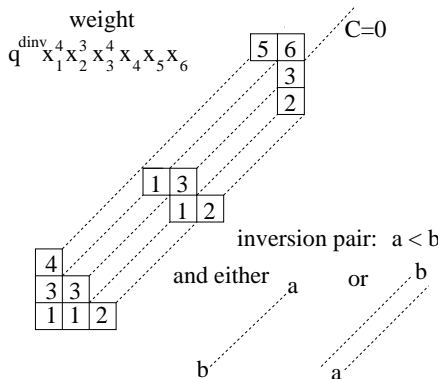
LLT Polynomials

In 1997, Lascoux, Leclerc and Thibon introduced ribbon tableaux generating functions, commonly known as LLT polynomials, which depend on a tuple of skew shapes, x_1, \dots, x_n and a parameter q . They proved these polynomials are symmetric in the x_i and conjectured they are Schur positive. (Now proved by Grojnowski-Haiman).

- In HHL we show how $\tilde{H}_\mu(X; q, t)$ can be expressed as a sum of LLT polynomials times nonnegative powers of t . Thus Schur positivity of $\tilde{H}_\mu(X; q, t)$ follows from Schur positivity of LLT polynomials. We also give a combinatorial proof that LLT polynomials are symmetric functions.

One Description of LLT's

The LLT polynomial equals a power of q times the sum, over all tuples \mathbf{T} of SSYT of skew shape, of $q^{\text{dinv}(\mathbf{T})}$, where $\text{dinv}(\mathbf{T})$ is the total number of inversion pairs, described below.

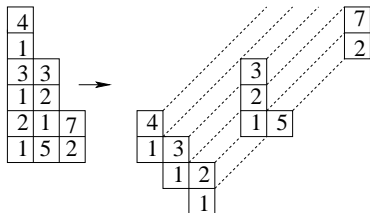


Theorem : If we fix a descent set D , then

$$\sum_{\sigma: \text{Des}(\sigma)=D} t^{\text{maj}(\sigma, \mu)} q^{\text{inv}(\sigma, \mu)} x^\sigma$$

is a positive multiple of an LLT product of ribbons.

Proof: There is a bijection between fillings with a fixed descent set D and tuples of SSYT of ribbons:



Now use the fact that

$$\text{inv}(\sigma, \mu) = \text{dinv} - \sum_{w \in \text{Des}(\sigma)} \text{arm}(w).$$

The Symmetry Problem: The definition of the \tilde{H}_μ implies

$$\tilde{H}_\mu(X; q, t) = \tilde{H}_{\mu'}(X; t, q),$$

where μ' is the conjugate partition. Can we understand this bijectively?

In the case where $\mu = 1^n$, this symmetry reduces to MacMahon's theorem, that maj and inv are equidistributed over multiset permutations. A bijective proof of this was given by Foata:

$$\text{maj}(3162475) = 1 + 3 + 6 = 10.$$

$$31 \rightarrow 31, \quad 316 \rightarrow 316, \quad 3162 \rightarrow 3|16|2 \rightarrow 3612,$$

$$36124 \rightarrow 3|61|2|4 \rightarrow 31624, \quad 361247 \rightarrow 361247,$$

$$3612475 \rightarrow 36|1247|5 \rightarrow 6371245 = \phi(3162475).$$

Theorem (Foata-Schützenberger)

Let β denote the inverse map on S_n . Then

$$\beta\phi\beta\phi^{-1}\beta$$

is an involution on S_n which interchanges maj and inv.

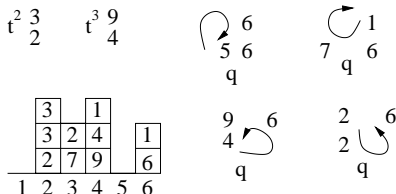
We would like a map γ_μ on S_n satisfying

$$\begin{aligned}\text{inv}(\gamma_\mu(\sigma), \mu') &= \text{maj}(\sigma, \mu) \\ \text{maj}(\gamma_\mu(\sigma), \mu') &= \text{inv}(\sigma, \mu).\end{aligned}$$

Then

$$\begin{aligned}\sum_{\sigma \in S_n} t^{\text{maj}(\sigma, \mu)} q^{\text{inv}(\sigma, \mu)} &= \sum_{\sigma \in S_n} t^{\text{inv}(\gamma_\mu(\sigma), \mu')} q^{\text{maj}(\gamma_\mu(\sigma), \mu')} \\ &= \sum_{\sigma \in S_n} t^{\text{inv}(\sigma, \mu')} q^{\text{maj}(\sigma, \mu')}.\end{aligned}$$

In 1995 Macdonald introduced polynomials $\mathcal{E}_\alpha(x_1, \dots, x_n; q, t)$, now known as *nonsymmetric Macdonald polynomials*. Here $\alpha_i \in \mathbb{N}$. Like the J_μ , the \mathcal{E}_α satisfy a multivariate orthogonality condition. They form a basis for the polynomial ring $\mathbb{Q}[x_1, \dots, x_n](q, t)$.



Theorem (HHL, 2006) For any weak composition α ,

$$\mathcal{E}_\alpha(x_1, \dots, x_n; q, t) = \sum_{\substack{\text{nonattacking fillings } (T, \alpha') \\ \text{basement} = \{n, n-1, \dots, 1\}}} x^T q^{\text{maj}(T, \alpha')} t^{\text{coinv}(T, \alpha')} \\ \times \prod_{\substack{w \in \alpha' \\ T(w) = T(\text{South}(w))}} (1 - q^{1+\text{leg}(w)} t^{1+\text{arm}(w)}) \prod_{\substack{w \in \mu' \\ T(w) \neq T(\text{South}(w))}} (1 - t).$$

Let $\epsilon_n = (1, 2, \dots, n)$ and $\overline{\epsilon}_n = (n, n-1, \dots, 1)$. The case $q = t = 0$, basement $\overline{\epsilon}_n$, gives a new formula for the Demazure character (key polynomial) $\mathcal{A}_\alpha^{\overline{\epsilon}_n}(x_1, \dots, x_n)$ and, changing the basement to ϵ_n , a corresponding formula for the Demazure atom (standard bases) $\mathcal{A}_\alpha^{\epsilon_n}(x_1, \dots, x_n)$. The combinatorics of Demazure characters and atoms was first studied by Lascoux and Schützenberger, motivated by their study of Schubert polynomials. In particular they showed that the Schubert polynomial can be expressed as a positive sum of Demazure characters, and gave a nice combinatorial description of the coefficients. It is known that

$$s_\lambda(X) = \sum_{\alpha \sim \lambda} \mathcal{A}_\alpha^{\epsilon_n}(x_1, \dots, x_n).$$

Sarah Mason gave a bijective proof of this, involving an extension of the RSK algorithm. Thank you!

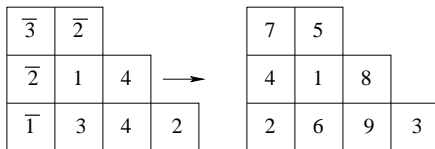
The Proof of the Combinatorial Formula for $\tilde{H}_\mu(\mathbf{X}; \mathbf{q}, \mathbf{t})$

Proposition

Let \mathcal{A} be an alphabet of positive and negative letters, with any fixed total ordering. Then

$$CH_\mu[X(w-1); q, t] = \sum_{\sigma: \mu \rightarrow \mathcal{A}} (-1)^{\#\text{neg}} w^{\#\text{pos}} q^{\text{inv}} t^{\text{maj}} x^{|\sigma|}$$

where $\#\text{neg}$, $\#\text{pos}$ are the $\#$ of negative and positive letters, respectively.



$$(-1)^4 t^4 w^5 q^4 x_1^2 x_2^3 x_3^2 x_4^2$$

$$1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}$$

Involution 1

(a) We say two squares “attack” each other if they are either in the same row, or in successive rows, with the square in the row above strictly to the right of the one in the row below. Find the last attacking pair of 1’s, $\bar{1}$ ’s in the reading word (if none, find last attacking pair of 2’s, etc.).

(b) Switch the sign of first element (in reading word order) of attacking pair.

(c) Use ordering $1 < \bar{1} < \dots < n < \bar{n}$.

- The Descent set is fixed, so the t -weight is fixed. The q -weight is also fixed
- The fixed points are those “nonattacking” super fillings without attacking pairs, so at most one $1, \bar{1}$ in any row, at most one $2, \bar{2}$ in any row, etc. Thus if the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \dots$ is nonzero, we must have $\lambda_1 \leq \mu'_1, \lambda_1 + \lambda_2 \leq \mu'_1 + \mu'_2$, etc.

Involution 2

- (a) Find 1st 1 or $\bar{1}$ in reading word, not in the bottom row. (if none, find first 2 or $\bar{2}$ not in the bottom two rows, etc.).
- (b) Switch the sign of this element.
- (c) Use the ordering

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

- The q and t weights are preserved.
- The fixed points have 1, $\bar{1}$'s only in the bottom row, 2, $\bar{2}$'s only in the bottom two rows, etc. Thus if coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ is nonzero, we must have $\lambda_1 \leq \mu_1$, $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$, etc.

The fact that $CH_\mu(1; q, t) = 1$ is easy to show, and the proof is complete.