

Noncommutative Schur functions

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March 21, 2016

Symmetric functions

Symmetric functions (in finitely or infinitely many variables) play important roles in various branches of mathematics.

Basic families of symmetric functions:

- elementary symmetric functions

$$e_k(\mathbf{x}) = e_k(x_1, x_2, \dots) = \sum_{i_1 > \dots > i_k} x_{i_1} \cdots x_{i_k};$$

- complete homogeneous symmetric functions

$$h_k(\mathbf{x}) = h_k(x_1, x_2, \dots) = \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \cdots x_{i_k};$$

- Schur functions $s_\lambda(\mathbf{x}) = s_\lambda(x_1, x_2, \dots)$.

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Schur functions

Schur functions s_λ are labeled by integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$.

Many alternative definitions, including:

- Cauchy-Jacobi bialternant formula;
- Jacobi-Trudi determinantal formula:

$$s_\lambda(\mathbf{x}) = \det(h_{\lambda_i - i + j}(\mathbf{x}))_{i,j=1}^{\ell(\lambda)};$$

- Kostka-Nagelsbach determinantal formula

$$s_\lambda(\mathbf{x}) = \det(e_{\lambda_i - i + j}(\mathbf{x}))_{i,j=1}^{\lambda_1};$$

- monomial expansion in terms of semistandard Young tableaux;
- axiomatic definitions based on orthogonality properties.

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Applications of Schur functions

Schur functions arise in many contexts, including:

- representation theory of symmetric groups \mathcal{S}_n ;
- representation theory of general linear groups $\mathrm{GL}_n(\mathbb{C})$;
- intersection theory on Grassmann manifolds $\mathrm{Gr}(k, n)$.

Fundamental Problem

Expand a given symmetric function in the basis of Schur functions.

Applications:

- expanding a representation of \mathcal{S}_n or $\mathrm{GL}_n(\mathbb{C})$ into irreducible modules;
- expanding a cohomology class of a subvariety in $\mathrm{Gr}(k, n)$ in the basis of Schubert cycles.

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The coefficients in these expansions are **positive**.

Schur positivity problems in algebraic combinatorics

Refinement of the Fundamental Problem

Find a manifestly positive combinatorial rule for the coefficients of the expansion of a given symmetric function in the basis of Schur functions.

Prototypical example: The Littlewood-Richardson Rule

Each of the many versions of this rule is a positive combinatorial formula for the Littlewood-Richardson coefficients $c_{\mu\nu}^{\lambda}$ defined by

$$s_{\mu}(\mathbf{x}) s_{\nu}(\mathbf{x}) = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}(\mathbf{x}).$$

These numbers arise, in particular:

- as tensor product multiplicities for irreducible representations of GL_n ;
- as multiplicities for the induction product for \mathcal{S}_n -modules;
- as intersection numbers for Schubert classes in $H^*(Gr(k, n))$.

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Important unsolved Schur positivity problems

Find a positive combinatorial formula for

- **Kronecker coefficients**: decomposition multiplicities of a tensor product of irreducible representations of the symmetric group.
- **Kostka-Macdonald coefficients**: coefficients in the Schur expansion of a transformed **Macdonald polynomial**;
- coefficients in the Schur expansions of **LLT polynomials**.

Some solved Schur positivity problems

Positive combinatorial formulas are known for the Schur expansions of

- LLT polynomials indexed by 3-tuples of skew shapes, and transformed Macdonald polynomials indexed by shapes with 3 columns [J. Blasiak];
- [Stanley symmetric functions](#), the stable limits of [Schubert polynomials](#) of Lascoux-Schützenberger [P. Edelman–C. Greene];
- [Stable Grothendieck polynomials](#), K -theoretic analogues of Stanley symmetric functions [SF–C. Greene].

All these results can be obtained using the machinery of *noncommutative Schur functions*.

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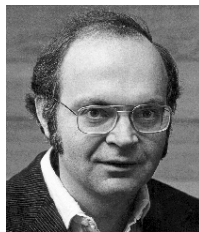
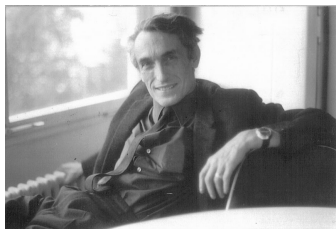
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Noncommutative Schur functions: early history

This approach can be viewed as a generalization of Schützenberger's celebrated proof of the Littlewood-Richardson Rule (using the **plactic monoid** defined by the **Knuth relations**).



M.-P. Schützenberger, La correspondance de Robinson, *Combinatoire et représentation du groupe symétrique*, Springer LNM **579** (1977), 59–113.

D. Knuth, Permutations, matrices, and generalized Young tableaux, *Pacific J. Math.* **34** (1970), 709–727.

Noncommutative e 's and h 's

$\mathcal{U} = \mathbb{Z}\langle u_1, \dots, u_N \rangle =$ free associative ring generated by $\mathbf{u} = (u_1, \dots, u_N)$

Noncommutative elementary symmetric functions:

$$e_k(\mathbf{u}) = \sum_{N \geq i_1 > i_2 > \dots > i_k \geq 1} u_{i_1} u_{i_2} \cdots u_{i_k}$$

Equivalently, $E(x) = \sum_{k=0}^N x^k e_k(\mathbf{u}) = (1 + xu_N) \cdots (1 + xu_1) \in \mathcal{U}[x]$.

Noncommutative complete homogeneous symmetric functions:

$$h_k(\mathbf{u}) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq N} u_{i_1} u_{i_2} \cdots u_{i_k}$$

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When the noncommutative e 's and h 's commute...

Lemma

For an ideal $I \subset \mathcal{U}$, the following are equivalent:

- the elements $e_k(\mathbf{u})$ commute pairwise modulo I ;
- the elements $h_\ell(\mathbf{u})$ commute pairwise modulo I .

If these conditions hold, then the product

$$H(x_1)H(x_2)\cdots \text{ mod } I$$

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From noncommutative to ordinary symmetric functions

\mathcal{U} has a basis of noncommutative monomials $u_{i_1} \cdots u_{i_m}$

\mathcal{U}^* has a basis of words in the alphabet $\{1, \dots, N\}$

An element of \mathcal{U}/I has well-defined pairings with vectors $\gamma \in I^\perp \subset \mathcal{U}^*$.

Assume the $e_k(\mathbf{u})$ commute pairwise modulo I .

For $\gamma \in I^\perp$, the formula

$$F_\gamma(x_1, x_2, \dots) = \langle H(x_1) H(x_2) \cdots, \gamma \rangle$$

defines an (ordinary) symmetric function in x_1, x_2, \dots .

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Many important families of symmetric functions arise this way:

- (skew) Schur functions;
- Stanley polynomials;
- stable Grothendieck polynomials;
- Macdonald polynomials;
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When do the $e_k(\mathbf{u})$ commute?

For $S \subset \{1, \dots, N\}$ and $k \leq |S|$, denote

$$e_k(\mathbf{u}_S) = \sum_{\substack{i_1 > i_2 > \dots > i_k \\ i_1, \dots, i_k \in S}} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

Theorem (J. Blasiak–SF–A. N. Kirillov)

For an ideal I of \mathcal{U} , the following are equivalent:

- *for any S and any $k, \ell \leq |S|$, we have*

$$e_k(\mathbf{u}_S) e_\ell(\mathbf{u}_S) \equiv e_\ell(\mathbf{u}_S) e_k(\mathbf{u}_S) \pmod{I};$$

- *these commutation relations hold for any S with $|S| \leq 3$.*

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Theorem (SF–C. Greene)

Let $I \subset \mathcal{U}$ be an ideal such that the $e_k(\mathbf{u})$ commute pairwise modulo I . Let $\gamma \in I^\perp$. Then the Schur expansion of F_γ is given by

$$F_\gamma(\mathbf{x}) = \sum_{\lambda} s_{\lambda}(\mathbf{x}) \langle s_{\lambda}(\mathbf{u}), \gamma \rangle$$

where

$$\begin{aligned} s_{\lambda}(\mathbf{u}) &= \det(e_{\lambda'_i - i + j}(\mathbf{u}))_{i,j=1}^{\lambda_1} \\ &= \sum_{\pi \in \mathcal{S}_{\lambda_1}} \operatorname{sgn}(\pi) e_{\lambda'_1 + \pi(1) - 1}(\mathbf{u}) \cdots e_{\lambda'_{\lambda_1} + \pi(\lambda_1) - \lambda_1}(\mathbf{u}). \end{aligned}$$

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The elements $s_{\lambda}(\mathbf{u}) \in \mathcal{U}$ are called the **noncommutative Schur functions**.

Schur positivity of symmetric functions F_γ

$$F_\gamma(\mathbf{x}) = \sum_{\lambda} s_\lambda(\mathbf{x}) \langle s_\lambda(\mathbf{u}), \gamma \rangle$$

$\mathcal{U}_{\geq 0} = \mathbb{Z}_{\geq 0}$ -cone in \mathcal{U} spanned by the noncommutative monomials

$\mathcal{U}_{\geq 0}^* = \mathbb{Q}_{\geq 0}$ -cone in \mathcal{U} spanned by words

In all applications, the vector $\gamma \in I^\perp$ lies in $\mathcal{U}_{\geq 0}^*$.

Theorem (J. Blasiak–SF)

For a homogeneous ideal $I \subset \mathcal{U}$ such that the $e_k(\mathbf{u})$ commute pairwise modulo I , the following are equivalent:

- all symmetric functions $F_\gamma(\mathbf{x})$, for $\gamma \in I^\perp \cap \mathcal{U}_{\geq 0}^*$, are Schur positive;*
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Schur positivity of $F_\gamma(\mathbf{x})$ vs. monomial positivity of $s_\lambda(\mathbf{u})$

Thus the following two problems are essentially equivalent to each other:

- finding a positive combinatorial formula for the Schur expansions of the symmetric functions $F_\gamma(\mathbf{x})$;
- finding an expansion of the noncommutative Schur functions $s_\lambda(\mathbf{u})$ as a positive linear combination of noncommutative monomials.

The switchboard ideal

The **switchboard ideal** $I_S \subset \mathcal{U}$ is generated by the elements

$$u_b^2 u_a + u_a u_b u_a - u_b u_a u_b - u_b u_a^2 \quad (b - a = 1),$$

$$u_b^2 u_a - u_b u_a u_b \quad (b - a \geq 2),$$

$$u_b u_a^2 - u_a u_b u_a \quad (b - a \geq 2),$$

$$u_b u_c u_a + u_a u_c u_b - u_b u_a u_c - u_c u_a u_b \quad (a < b < c).$$

Corollary

The $e_k(\mathbf{u})$ commute pairwise modulo I_S .

All ideals I arising in important applications contain I_S . The corresponding symmetric functions F_γ are labelled by nonnegative vectors

$$\gamma \in I^\perp \subset I_S^\perp.$$

The switchboard ideal

The **switchboard ideal** $I_S \subset \mathcal{U}$ is generated by the elements

$$\begin{aligned}u_b^2 u_a + u_a u_b u_a - u_b u_a u_b - u_b u_a^2 & \quad (b - a = 1), \\u_b^2 u_a - u_b u_a u_b & \quad (b - a \geq 2), \\u_b u_a^2 - u_a u_b u_a & \quad (b - a \geq 2), \\u_b u_c u_a + u_a u_c u_b - u_b u_a u_c - u_c u_a u_b & \quad (a < b < c).\end{aligned}$$

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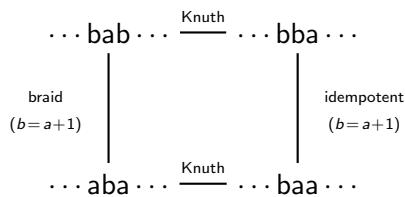
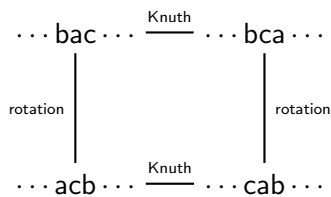
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Switchboards

Two words of the same length are related by a *switch* in position i if

- their entries match to the left of position $i-1$ and to the right of $i+1$;
- their entries in positions $i-1, i, i+1$ are related in one of these ways:



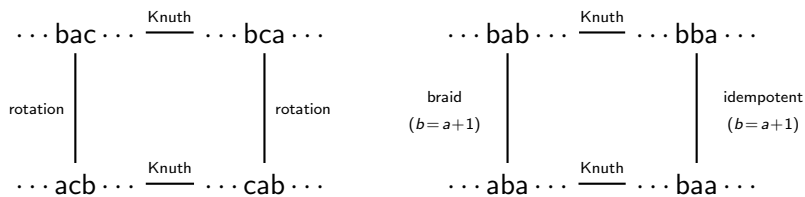
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- each edge labeled i corresponds to a switch in position i ;
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Switchboards and the switchboard ideal

Proposition

For a $(0, 1)$ -vector $\gamma \in \mathcal{U}^*$ the following are equivalent:

- $\gamma \in I_S^\perp$;
- γ is the sum of words appearing in some switchboard Γ .

We thus focus on the study of symmetric functions

$$F_\Gamma(\mathbf{x}) = F_\gamma(\mathbf{x})$$

associated to various classes of switchboards.

Alternative description of F_Γ in terms of **quasisymmetric functions**:

$$F_\Gamma(\mathbf{x}) = \sum_{w_1 \cdots w_n \in \text{Vert}(\Gamma)} \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_n \\ w_j > w_{j+1} \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

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Different switchboards on the same set of vertices

$$2134 \xrightarrow{2} 2314 \xrightarrow{3} 2341$$

$$2143 \xrightleftharpoons[3]{2} 2413$$

all edges are Knuth switches

$$\begin{array}{ccc} 2134 & \xrightarrow{2} & 2314 & & 2341 \\ & & \downarrow \bar{3} & & \downarrow \bar{3} \\ & & 2143 & \xrightarrow{2} & 2413 \end{array}$$

2-edges are Knuth switches

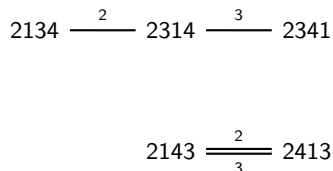
3-edges are rotation switches

For each of these two switchboards, we have

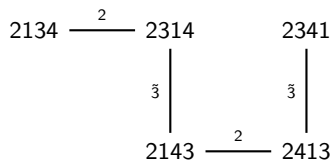
$$F_{\Gamma}(\mathbf{x}) = s_{31}(\mathbf{x}) + s_{22}(\mathbf{x}).$$

Connected components of a switchboard are switchboards.

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The switchboard on \mathcal{S}_4 with only Knuth switches

2134 $\xrightarrow{2}$ 2314 $\xrightarrow{3}$ 2341 3421 $\xrightarrow{2}$ 3241 $\xrightarrow{3}$ 3214

3124 $\xrightarrow{2}$ 1324 $\xrightarrow{3}$ 1342 2431 $\xrightarrow{2}$ 4231 $\xrightarrow{3}$ 4213

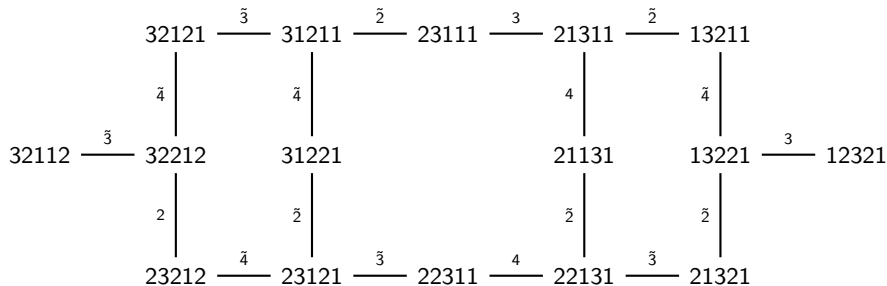
4123 $\xrightarrow{2}$ 1423 $\xrightarrow{3}$ 1243 1432 $\xrightarrow{2}$ 4132 $\xrightarrow{3}$ 4312

1234 2413 $\xrightarrow[3]{2}$ 2143 4321

3412 $\xrightarrow[3]{2}$ 3142

$$F_{\Gamma} = s_4 + 3s_{31} + 2s_{22} + 3s_{211} + s_{1111}$$

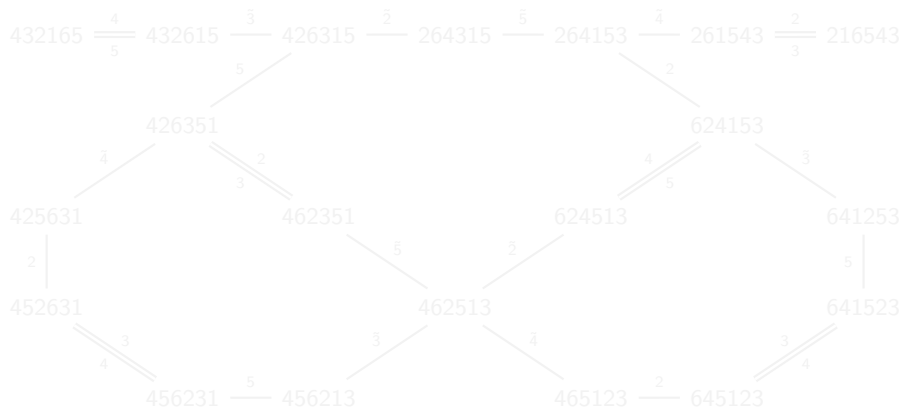
Another switchboard and associated symmetric function



$$F_{\Gamma} = s_{32} + s_{311} + s_{221}$$

Switchboards and Schur positivity

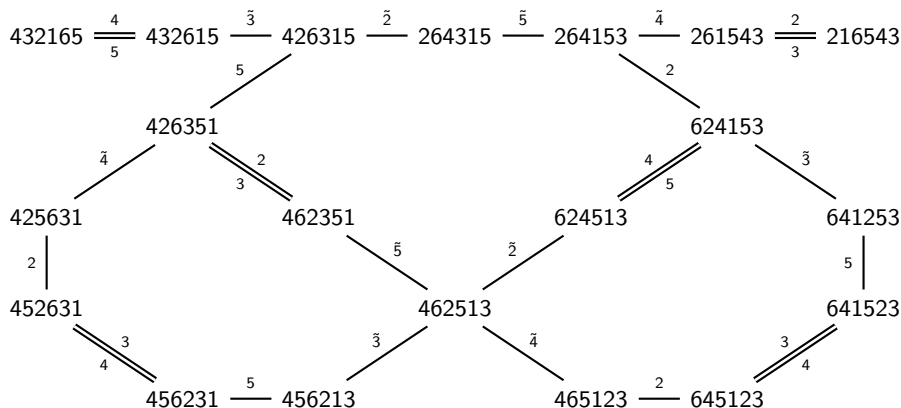
Are all symmetric functions F_{Γ} associated to switchboards Schur positive?



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Switchboards and Schur positivity

Are all symmetric functions F_Γ associated to switchboards Schur positive?



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Schur positivity of symmetric functions F_Γ

Big questions (essentially equivalent to each other):

- For which classes of switchboards Γ is $F_\Gamma(\mathbf{x})$ Schur positive?
- For which ideals $I \supset I_S$ is $F_\gamma(\mathbf{x})$ Schur positive for $\gamma \in I^\perp \cap \mathcal{U}_{\geq 0}^*$?
- For which ideals $I \supset I_S$ do the noncommutative Schur functions $s_\lambda(\mathbf{u})$ have monomial positive expressions modulo I ?

The plactic ideal

The **plactic ideal** $I_{\text{plac}} \subset \mathcal{U}$ is generated by the elements

$$u_a u_c u_b - u_c u_a u_b \quad \text{for } a \leq b < c;$$

$$u_b u_a u_c - u_b u_c u_a \quad \text{for } a < b \leq c.$$

Theorem (M.-P. Schützenberger)

Modulo the plactic ideal, each noncommutative Schur function $s_\lambda(\mathbf{u})$ can be written as the sum of noncommutative monomials corresponding to column readings of semistandard tableaux of shape λ .

$(0, 1)$ -vectors in $I_{\text{plac}}^\perp \longleftrightarrow$ switchboards whose all switches are Knuth

Theorem (I. Gessel)

Let Γ be a connected switchboard with only Knuth switches. Then $F_\Gamma(\mathbf{x})$ is a Schur function.

Conclusion: the plactic ideal is too large.

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Schützenberger's monomial expansion for noncommutative Schur functions holds modulo the nilCoxeter ideal.

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They are labeled by tuples of skew shapes, and involve a parameter q .

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LLT polynomials as symmetric functions F_γ

S. Assaf introduced a class of switchboards whose associated symmetric functions include LLT polynomials. We translate Assaf's construction into the language of noncommutative rings. (Cf. earlier work by T. Lam.)

For k a positive integer, let $I_{\mathbb{R},k}$ denote the ideal in \mathcal{U} generated by

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Theorem

The class of symmetric functions F_γ labeled by vectors $\gamma \in I_{\mathbb{R},k}^\perp \cap \mathcal{U}_{\geq 0}^$ includes the coefficients of powers of q in arbitrary LLT polynomials.*

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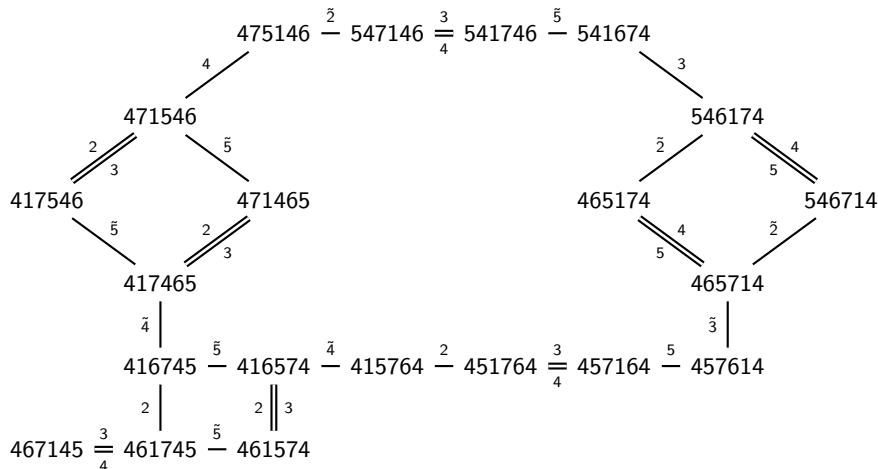
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LLT switchboards



$$F_{\Gamma} = s_{33} + s_{321} = \text{coefficient of } q^3 \text{ in the LLT polynomial indexed by } (3/2, 33/11, 2/1)$$

Towards a solution of the LLT positivity problem

Conjecture

For any λ and any $k > 0$, we have $s_\lambda(\mathbf{u}) \in \mathcal{U}_{\geq 0} + I_{\mathbb{R},k}$.

Consequently $F_\gamma(\mathbf{x})$ is Schur positive for any $\gamma \in \mathcal{U}_{\geq 0}^* \cap I_{\mathbb{R},k}$.

A formula representing the noncommutative Schur function $s_\lambda(\mathbf{u})$ as a nonnegative combination of noncommutative monomials (modulo $I_{\mathbb{R},k}$) would yield a manifestly positive combinatorial rule for the coefficients in the Schur expansion of an LLT polynomial.

For LLT polynomials labeled by 3 skew shapes, such a formula for $s_\lambda(\mathbf{u})$ was recently obtained by J. Blasiak.

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