

Schützenberger's jeu de taquin

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Conference dedicated to
the scientific legacy of Marcel-Paul Schützenberger
Bordeaux, March 22nd, 2016

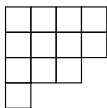


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Outline

- 1 Jeu de taquin
- 2 Knuth equivalence and Lascoux-Schützenberger's plactic monoid
- 3 Motivation: product of Schur functions
- 4 Jeu de taquin in recent research work

(Straight) Young diagram: set of boxes packed in the north-west corner.

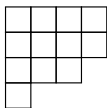


(Straight) Young tableau: filling of a Young diagram

- weakly increasing along rows;
- strictly increasing along columns.

1	1	2	4
2	3	3	5
4	4	5	
6			

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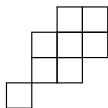


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Skew diagram: difference set between two Young diagrams



Skew tableau: filling of a skew diagram with the same conditions

		2	4
	3	3	5
	4	5	
2			

Jeu de taquin

Start with a skew tableau and color in red one of its *inner corner*.

	2	4
3	3	5
4	5	
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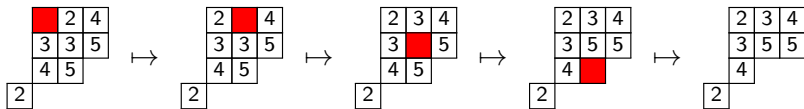
Move the red box according to the following rule:

$$\begin{array}{|c|c|} \hline \text{red} & x \\ \hline y & \\ \hline \end{array} \mapsto \begin{cases} \begin{array}{|c|c|} \hline x & \text{red} \\ \hline y & \\ \hline \end{array} & \text{if } x < y; \\ \\ \begin{array}{|c|c|} \hline y & x \\ \hline \text{red} & \\ \hline \end{array} & \text{if } y \leq x. \end{cases}$$

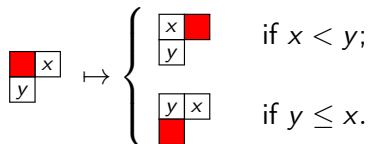
until the red box is outside the diagram; then erase the red box.

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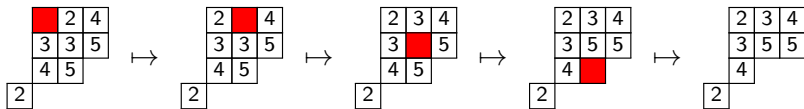
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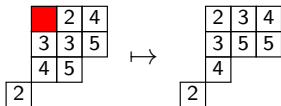
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→ The rule ensures that we keep a tableau.

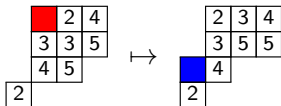
Rectification of a skew-tableau

Start with a skew tableau T . Successively choose inner corners and apply jeu de taquin until you get a tableau.



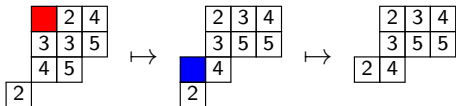
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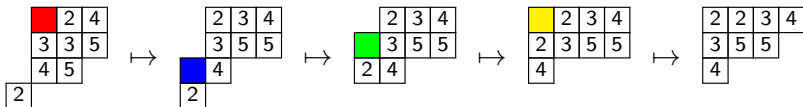
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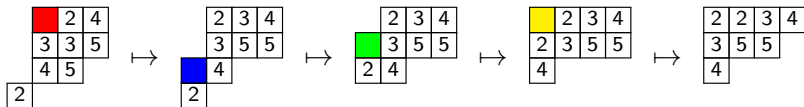
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Theorem

The final straight tableau does not depend on the order in which we choose inner corners.

This final tableau is called the *rectification* of T and denoted $\text{Rect}(T)$.

Reading word of a skew tableau (from bottom to top, from left to right):

$$T = \begin{array}{c} \begin{array}{cc} 2 & 4 \\ 3 & 3 & 5 \\ 4 & 5 \end{array} \\ 2 \end{array} \mapsto w(T) = 24533524.$$

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$$T = \begin{array}{c} & & 2 & 4 \\ & 3 & 3 & 5 \\ & 4 & 5 & \\ 2 & & & \end{array} \mapsto w(T) = 24533524.$$

Definition

Two words are Knuth equivalent if one can be obtained from the other by *elementary Knuth transforms*

$$y z x \equiv y x z \text{ (if } x < y \leq z), \quad x z y \equiv z x y \text{ (if } x \leq y < z).$$

$$\begin{array}{|c|c|} \hline x & \color{red} \\ \hline y & z \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline x & z \\ \hline y & \color{red} \\ \hline \end{array}$$

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Lemma

If one obtains T' from T by jeu de taquin, then $w(T)$ and $w(T')$ are Knuth equivalent.

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The monoid of words up to Knuth equivalence is the *plactic monoid*, studied by Lascoux and Schützenberger.

Uniqueness of the rectification

Theorem

Each Knuth equivalence class contains the reading word of a unique straight tableau.

Corollary

$\text{Rect}(T)$ is the unique straight tableau whose reading word is in the same class as $w(T)$ (and thus does not depend on the successive choices of corners).

If T is a tableau, write $x^T = x_1^{\#1 \text{ in } T} \cdot x_2^{\#2 \text{ in } T} \dots$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & 5 \\ \hline 4 & 5 & 5 & \\ \hline 6 & & & \\ \hline \end{array}, \quad x^T = x_1^2 x_2^2 x_3^3 x_4 x_5^3 x_6.$$

Then if λ is a diagram, the associated *Schur function* is

$$s_\lambda := \sum_T x^T,$$

where the sum is taken over all tableaux of shape λ .

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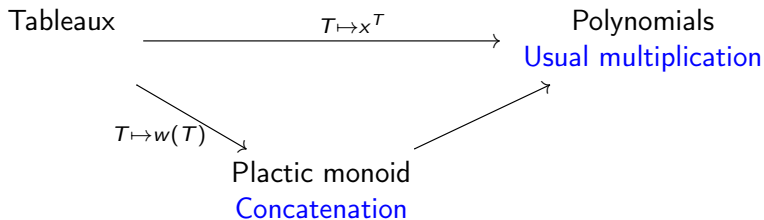
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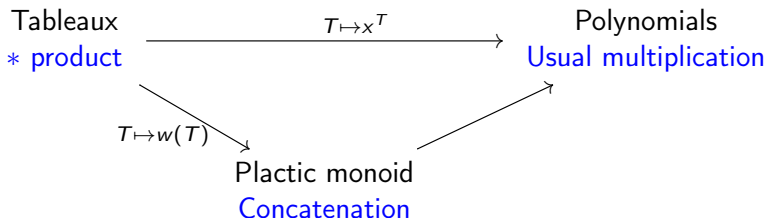
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Problem

Describe *Littlewood-Richardson coefficients* $c_{\mu,\nu}^\lambda$ defined by

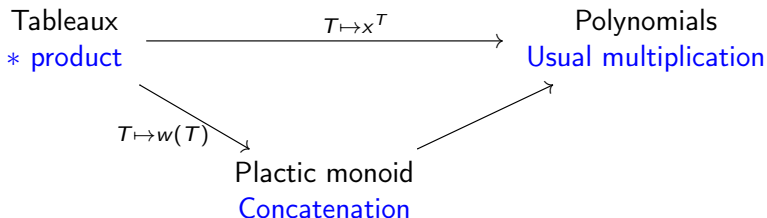
$$s_\mu s_\nu = \sum_\lambda c_{\mu,\nu}^\lambda s_\lambda.$$





Definition

$$T * U := \text{Rect} \left(\begin{array}{|c|c|c|} \hline \square & U & \\ \hline T & & \\ \hline \end{array} \right). \text{ Example: } \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 4 & \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \text{Rect} \left(\begin{array}{|c|c|c|} \hline & & 1 & 3 \\ & & 2 & \\ \hline 1 & 2 & 2 & \\ \hline 2 & 4 & & \\ \hline \end{array} \right).$$

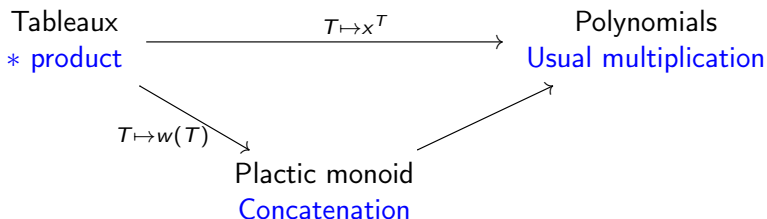


Proposition

Denote $S_\mu = \sum_{\substack{T \text{ of} \\ \text{shape } \mu}} T$. In the algebra of tableaux

$$S_\mu * S_\nu = \sum_{\lambda} c_{\mu,\nu}^\lambda S_\lambda.$$

→ enables to describe combinatorially products of Schur functions.



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Remarkable fact: the algebra of the plactic monoid contains the symmetric function ring as a subalgebra.

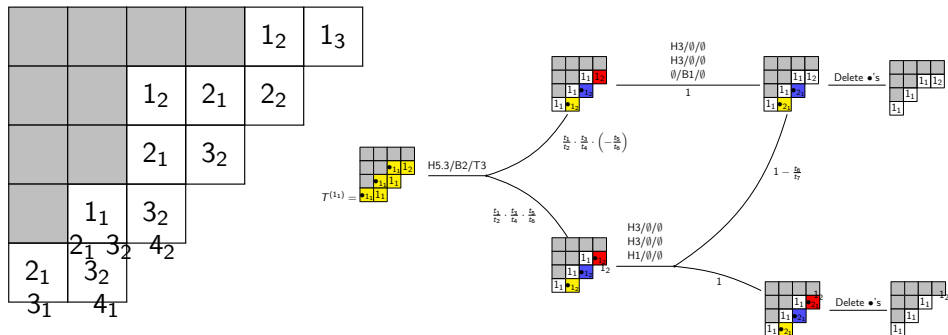
Jeu de taquin in recent research work

- 1 in algebraic combinatorics (Pechenik, Yong 2015);
- 2 in bijective combinatorics (Fang 2015);
- 3 in probabilistic combinatorics (Romik, Śniady 2015).

Pechenik-Yong's combinatorial rule

- Littlewood-Richardson coefficients also appear in the “cohomology ring” of the Grassmannian.
- Pechenik and Yong (2015) consider an analogue of Littlewood-Richardson coefficients in the “equivariant K-theory” of the Grassmannian.
- They find a combinatorial rule using enriched tableaux and adapting jeu de taquin to these tableaux.

Pechenik-Yong's tableaux and jeu de taquin



©Pechenik-Yong

→ jeu de taquin is a robust technique!

Bijjective proof of character identities (Fang)

Denote f^λ the number of standard tableaux of shape λ .

Theorem (Frobenius, Murnaghan-Nakayama)

$$2 f^\lambda \left(\sum_{(i,j) \in \lambda} (j - i) \right) = \binom{n}{2} (f^{\lambda/(2)} - f^{\lambda/(1,1)}).$$

Both sides compute symmetric group characters on transpositions. . .

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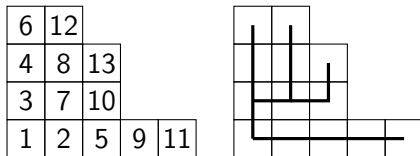
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Fang (2015) gave a bijective proof using jeu de taquin



©Fang

Jeu de taquin on random infinite tableaux (Romik, Śniady)

		⋮		⋯	
10	12	38	39	40	
7	11	23	36	37	
5	8	15	24	28	⋯
3	6	9	18	20	
1	2	4	13	16	

A random infinite tableau.

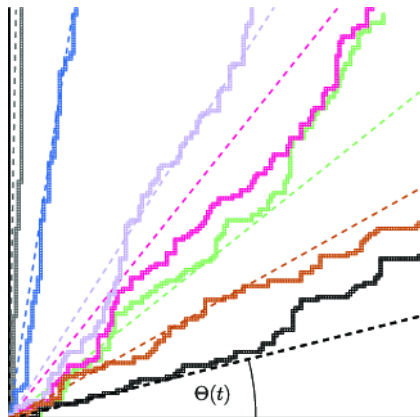
©Romik, Śniady

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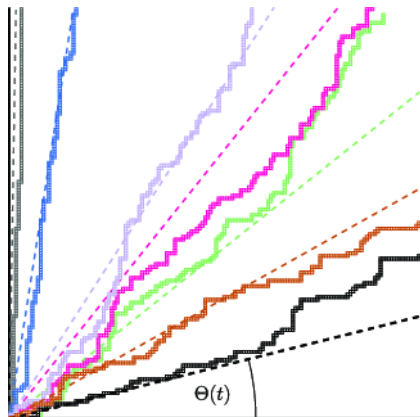


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©Romik, Śniady



Romik and Śniady (2015) show that the jeu de taquin paths are

- almost surely asymptotically straight lines,
- with a random direction whose distribution is explicit.