

Algebraic characterization of logics on countable linear orderings

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Colloque en l'honneur de Marcel-Paul Schützenberger

Based on works of N. Bedon, A. Bès, V. Bruyère, Th. Colcombet,
G. Puppis, C. Rispal, G. Sénizergues and A.V. Sreejith

Outline

History

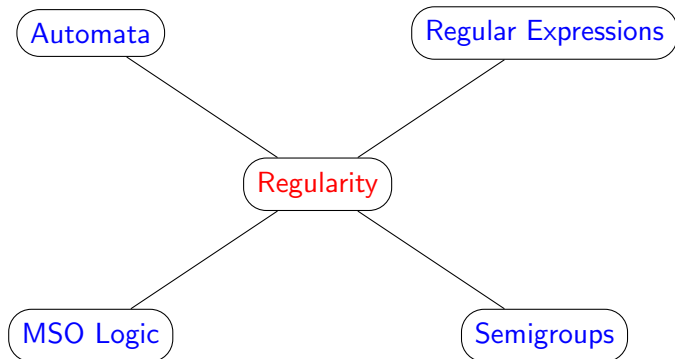
Words

MSO

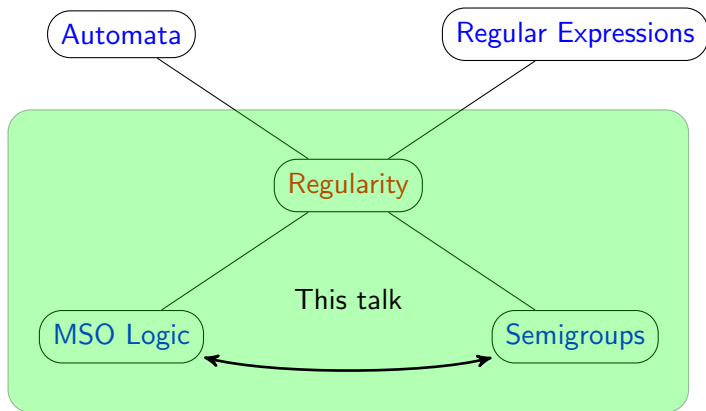
Algebras

Characterizations of fragments of MSO

Formalisms for regularity



Formalisms for regularity



History

MSO = Regularity (finite words)
[Elgot - Büchi 60]

$(\mathbb{N}, <)$ [Büchi 62]

$(\omega_1, <)$ [Büchi 65]

$(\mathbb{Q}, <)$ [Rabin 69]

$(\mathbb{Q}, <)$ [Shelah 75]

$(\mathbb{R}, <)$ [Shelah 75] (undecidable)

Scattered orderings
[Bruyère-C. 08]

Countable orderings
[Colcombet-C.-Puppis 11]

Extensions of the model

FO-definable = aperiodic
[Schützenberger 65]
[McNaughton-Papert 71]

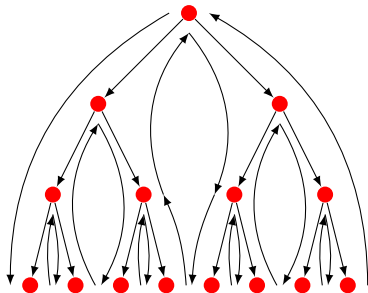
FO²-definable = DA
[Schützenberger 76]
[Thérien-Wilke 98]

Many logics

Restriction of the logic

?

Application of Rabin's theorem to $(\mathbb{Q}, <)$



Theorem (Rabin 1969)

The Monadic Second Order logic of $(\mathbb{Q}, <)$ is decidable.

This gives:

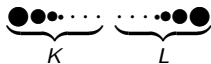
- ▶ No **good acceptors**.
- ▶ No **regular models**.
- ▶ No **collapse** of MSO

Cuts, gaps and completeness

- ▶ A **cut** of an ordering J is a partition of J in K and L where K is an initial interval and L is a final interval.



- ▶ A cut (K, L) is a **gap** if K has no greatest element and L has no least element.












- ▶ An ordering is **complete** if it has no gap.
 - ▶ \mathbb{Q} is not complete
 - ▶ \mathbb{R} is complete

Labelled Linear orderings (Words)

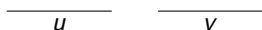
A **labelled linear ordering** is a linear ordering $(J, <)$ with a labeling function from J into a finite set $A = \{\bullet, \bullet\}$ of colors. The set of all countable words is A° .

Examples

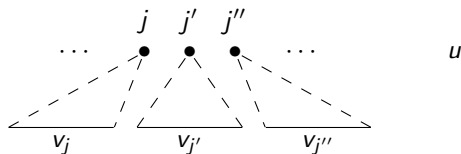
Word	Length
	3
	ω
	$\omega + \omega$
	$\tilde{\omega}$
	$\zeta = \tilde{\omega} + \omega$
	$\omega + \tilde{\omega}$
	ω^2
	η
	λ

Operations on words

- ▶ Mirror image: \tilde{u} backwards linear ordering
- ▶ Concatenation of words
 - ▶ $u \cdot v$ places v to the right of u



- ▶ Generalized concatenation: flattening from $(A^\circ)^\circ$ to A°
 - ▶ $\prod_{j \in J} v_j$ generalized sum



Scattered orderings and perfect shuffle

- ▶ **dense** ordering: $\forall i < j \exists k \quad i < k < j$.
- ▶ **scattered** ordering: no dense subordering.

Theorem (Hausdorff)

The class of countable scattered orderings is $\bigcup_{\alpha \in \mathcal{O}} V_\alpha$ where

- ▶ $V_0 = \{\mathbf{0}, \mathbf{1}\}$,
 - ▶ $V_\alpha = \{\sum_{j \in J} K_j \mid J \in \{\omega, \tilde{\omega}\} \text{ and } K_j \in \bigcup_{\beta < \alpha} V_\beta\}$.
-
- ▶ The **perfect shuffle** $\{\bullet, \bullet\}^\eta$ is the unique word such that \bullet and \bullet are dense.



$$\begin{cases} r \mapsto \bullet & \text{if } r = n/2^m \\ r \mapsto \bullet & \text{if } r \in \mathbb{Q} \setminus \{n/2^m \mid m \geq 0\} \end{cases}$$

Monadic Second Order Logic (MSO)

Logic with

- ▶ order $<$ and unary predicates $\bullet(x)$, $\bullet(x)$,
- ▶ Boolean connectives \vee , \wedge , \neg ,
- ▶ quantifications on elements $\exists x$ and $\forall x$,
- ▶ quantifications on sets of elements $\exists X$ and $\forall X$.

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Examples (Without labels)

- ▶ $(E, <)$ is **finite** iff it satisfies
 $\neg \exists X (\forall x \in X \exists y \in X \ x < y) \vee (\forall x \in X \exists y \in X \ x > y)$

Monadic Second Order Logic (MSO)

Logic with

- ▶ order $<$ and unary predicates $\bullet(x)$, $\bullet(x)$,
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Examples (Without labels)

- ▶ $(E, <)$ is **dense** iff it satisfies
 $(\exists x \exists y \ x < y) \wedge (\forall x \forall y \ \exists z \ x < z < y)$

Monadic Second Order Logic (MSO)

Logic with

- ▶ order $<$ and unary predicates $\bullet(x)$, $\bullet(x)$,
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Examples (Without labels)

- ▶ $(E, <)$ is **scattered** (no dense subordering) iff it satisfies $\neg \exists X \forall x \forall y \in X (x < y \implies \exists z \in X x < z < y)$

Monadic Second Order Logic (MSO)

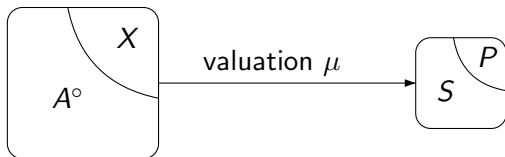
Logic with

- ▶ order $<$ and unary predicates $\bullet(x)$, $\bullet(x)$,
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- ▶ quantifications on elements $\exists x$ and $\forall x$,
- ▶ quantifications on sets of elements $\exists X$ and $\forall X$.

Example (With labels)

- ▶ blue points are dense iff
 $(\exists x \exists y \ x < y) \wedge (\forall x \forall y \ \exists z \ x < z < y \wedge \bullet(z))$

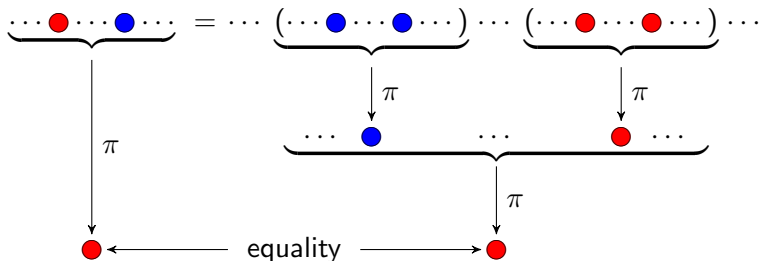
Algebras with an infinitary product π



$$x \in X \iff \mu(x) \in P$$

A **\circ -monoid** is a pair (S, π) where $\pi : S^\circ \rightarrow S$ satisfies

- ▶ $\pi(s) = s$,
- ▶ generalized associativity



Examples

Example (Finite words)

The set X of scattered words is recognizable. Consider $S = \{1, a, 0\}$ with the product defined by

$$\pi(x) = \begin{cases} 1 & \text{if } x \text{ is empty or contains only } 1 \\ a & \text{if } x \text{ has one but finitely many } a \text{ ans no } 0 \\ 0 & \text{otherwise} \end{cases}$$

Example (Scattered words)

The set X of scattered words is recognizable. Consider $S = \{1, a, 0\}$ with the product defined by

$$\pi(x) = \begin{cases} 1 & \text{if } x \text{ is empty or contains only } 1 \\ a & \text{if } x \text{ is scattered and contains only } 1 \text{ and } a \\ 0 & \text{otherwise} \end{cases}$$

Finite description of the product

Problem

The product π of a \circ -semigroup S may be difficult to describe, even when S is finite.

Define the operations

$$\begin{aligned} \bullet \cdot \bullet &= \pi(\bullet \bullet) & \bullet^\omega &= \pi(\bullet \bullet \dots) & \bullet^{\tilde{\omega}} &= \pi(\dots \bullet \bullet) \\ \{\bullet, \bullet, \dots\}^\eta &= \pi(\dots \bullet \bullet \dots \bullet \bullet \dots) \end{aligned}$$

Example (Scattered words)

$$a \cdot a = a \quad a \cdot 0 = 0 \cdot a = 0 \cdot 0 = 0$$

$$a^\omega = a^{\tilde{\omega}} = a \quad 0^\omega = 0^{\tilde{\omega}} = 0$$

$$\{a\}^\eta = \{0\}^\eta = \{a, 0\}^\eta = 0$$

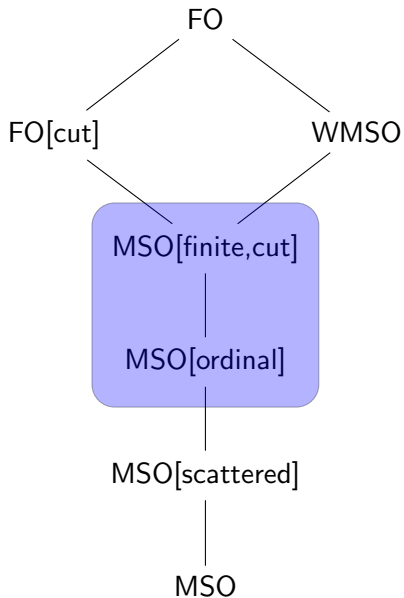
Proposition

The four operations fully describe the product π .

Restricting set quantifications

Range	Name of the logic
singleton sets	first-order: FO <i>is dense, has length k</i>
cuts	first-order on cuts: FO[cut] <i>is finite, is well ordered, is complete</i>
finite sets	weak monadic second-order logic: WMSO <i>is finite, has even length</i>
finite sets and cuts	MSO[finite, cut] <i>has an even number of gaps</i>
well ordered sets	MSO[ordinal] ...
scattered sets	MSO[scattered] <i>is scattered</i>
all sets	MSO <i>there are two sets dense in each other</i>

Structure



Can we separate these logics ?

Can we characterize these logics ?

Characterization of FO for scattered words

- ▶ An **idempotent** is an element e satisfying $e^2 = e$.

Theorem (Bès-C.)

For a set X of scattered words, the following conditions are equivalent.

- ▶ X is FO-definable.
- ▶ $M(X)$ is aperiodic and each idempotent e satisfies $e^\omega e^{\tilde{\omega}} = e$.
- ▶ X is marked star-free, that is, obtained from the letters by Boolean operations and marked product $X, Y \mapsto XaY$.

FO logic cannot detect gaps since it cannot separate

$$\underbrace{\bullet \bullet \dots \dots \bullet \bullet}_{\omega + \tilde{\omega}} \quad \text{and} \quad \underbrace{\bullet \bullet \dots \bullet}_{n} \quad \text{for } n \text{ large enough.}$$

Characterization of FO[cut] for scattered words

Theorem (Bedon-Rispa)

For a set X of scattered words, the following conditions are equivalent.

- ▶ *X is FO[cut]-definable.*
- ▶ *$M(X)$ is aperiodic.*
- ▶ *X is star-free, that is, obtained from the letters by Boolean operations and product $X, Y \mapsto XY$.*

Quantifying over cuts allows to describe a non marked product XY .

$$\underbrace{\quad}_{\in X} \mid \underbrace{\quad}_{\in Y} \quad w \in XY.$$

Characterization of WMSO for scattered words

- ▶ An ω -idempotent is an idempotent e satisfying $e^\omega = e$.
- ▶ An $\tilde{\omega}$ -idempotent is an idempotent e satisfying $e^{\tilde{\omega}} = e$.

Theorem (Bès-C., Colcombet-Sreejith)

For a set X of scattered words, the following conditions are equivalent.

- ▶ X is WMSO-definable.
- ▶ each ω -idempotent and each $\tilde{\omega}$ -idempotent satisfies $e^\omega e^{\tilde{\omega}} = e$.
- ▶ X can be obtained from the letters by Boolean operations, marked product $X, Y \mapsto XaY$ and marked star.

WMSO logic cannot detect gaps when in an infinite situation.

Characterization of FO[finite,cut] and FO[ordinal]

- ▶ An $\omega\tilde{\omega}$ -idempotent is an idempotent e satisfying $e^\omega = e^{\tilde{\omega}} = e$.
- ▶ An η -idempotent is an idempotent e satisfying $e^\eta = e$.
- ▶ An η -idempotent e is called **simple** if for all $K \subseteq M$ such that $eae = e$ for all $a \in M$, $(\{e\} \cup M)^\eta = e$.

Theorem (Colcombet-Sreejith)

For a set X of words, the following conditions are equivalent.

- ▶ X is FO[finite,cut]-definable.
- ▶ X is FO[ordinal]-definable.
- ▶ each $\omega\tilde{\omega}$ -idempotent is an η -idempotent and each η -idempotent is simple.

Characterization of FO[scattered]

Theorem (Colcombet-Sreejith)

For a set X of words, the following conditions are equivalent.

- ▶ *X is FO[scattered]-definable.*
- ▶ *each η -idempotent is simple.*

Merci