On algorithmic aspects of infinite games played on weighted graphs

Véronique Bruyère
UMONS Belgium

MPS 2016
1. Topic

2. Qualitative two-player zero-sum games

3. Quantitative two-player zero-sum games

4. Multidimensional two-player zero-sum games

5. Multiplayer non zero-sum games

6. Conclusion
This talk

- Focus on algorithmic game theory for the synthesis of correct computer systems
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- Some classical results and recent UMONS results
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- Some classical results and recent UMONS results
Algorithmic game theory

Reactive system

- System embedded into an uncontrollable environment
- It must satisfy some property against any behavior of the environment
- How to automatically design a correct controller for the system?
Algorithmic game theory

Reactive system

- System embedded into an uncontrollable environment
- It must satisfy some property against any behavior of the environment
- How to automatically design a correct controller for the system?

Example

- System: airplane and environment: weather
- The airplane must land safely in any weather conditions
- How to design a correct autopilot?
Algorithmic game theory

Reactive system

- System embedded into an uncontrollable environment
- It must satisfy some property against any behavior of the environment
- How to automatically design a correct controller for the system?

Modelization

- Two-player zero-sum game played on a finite directed graph
- Property = objective for the system
- Synthesis of a controller = construction of a winning strategy
1 Topic

2 Qualitative two-player zero-sum games

3 Quantitative two-player zero-sum games

4 Multidimensional two-player zero-sum games

5 Multiplayer non zero-sum games

6 Conclusion
Basic model

Definition

Two-player zero-sum game $G = (V, V_1, V_2, E, v_0)$:

- $(V, E)$ finite directed graph (with no deadlock)
- $(V_1, V_2)$ partition of $V$
- $V_p$ controlled by player $p \in \{1, 2\}$
- Initial vertex $v_0$

Paths

Play: infinite path from $v_0 = \rho_0 \rho_1 \ldots \in V^\omega$ in $G$

History: prefix $h$ of a play

Unravelling of $G$: infinite tree of all paths from $v_0$
Basic model

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- **Unravelling** of $G$: infinite tree of all paths from $v_0$
Basic model

**Objective:** set $\Omega \subseteq V^\omega$ of plays

**Zero-sum game:**
- objective $\Omega$ for player 1
- opposite objective $V^\omega \setminus \Omega$ for player 2
Basic model

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Definition

Given a set $U \subseteq V$, classical qualitative objectives are:
- Reachability objective: visit a vertex of $U$ at least once
- Büchi objective: visit a vertex of $U$ infinitely often
Basic model

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- **Reachability** objective: visit a vertex of $U$ at least once
- **Büchi** objective: visit a vertex of $U$ infinitely often

Given a coloring $c : V \rightarrow \{0, 1, \ldots, C\}$

- **Parity** objective: the maximum color seen infinitely often is even
Strategies

Strategy for player $p$:
function $\sigma : V^* V_p \rightarrow V$ such that
$\sigma(hv) = v'$ with $(v, v') \in E$
Strategies

Strategy for player $p$:
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Memoryless: when $\sigma(hv) = \sigma(v)$
Strategies

Strategy for player $p$:
function $\sigma : V \times V_p \rightarrow V$ such that $\sigma(hv) = v'$ with $(v, v') \in E$

Memoryless: when $\sigma(hv) = \sigma(v)$

Finite-memory: when $\sigma$ is recorded by a finite automaton
Strategies

Winning strategy for player $p$: ensure his objective against any strategy of the other player

A game is determined from initial vertex $v_0$ when

- either player 1 is winning for $\Omega$ from $v_0$
- or player 2 is winning for $V^\omega \setminus \Omega$ from $v_0$
Strategies

**Winning strategy for player p:** ensure his objective against any strategy of the other player

A game is **determined** from initial vertex $v_0$ when

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**Example**

- Objective of player 1: visit $v_3$
- Player 1 is (trivially) winning from $v_3$
- Player 2 is winning from $v_0$, $v_1$, and $v_2$

Memoryless strategy: looping on $v_2$
Strategies

Winning strategy for player $p$: ensure his objective against any strategy of the other player

A game is determined from initial vertex $v_0$ when

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- or player 2 is winning for $V^\omega \setminus \Omega$ from $v_0$

Another example

Parity game: Player 1 is winning from every vertex with a memoryless strategy

- Either player 2 eventually stays at $v_2$ $\rightarrow$ max color seen infinitely often $= 0$
- Or he infinitely often visits $v_3$ $\rightarrow$ max color seen infinitely often $= 2$
Martin’s theorem

**Theorem [Mar75]**

Every game with Borel objectives is determined
Martin’s theorem

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Every game with Borel objectives is determined

- Need of the axiom of choice to exhibit a non-determined game
- No information about the winning strategies
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- Need of the **axiom of choice** to exhibit a non-determined game
- **No** information about the winning strategies

**Corollary**

Every game with \( \omega \)-regular objectives is determined
Martin’s theorem

**Theorem [Mar75]**

Every game with **Borel** objectives is determined

- Need of the **axiom of choice** to exhibit a non-determined game
- **No** information about the winning strategies

**Corollary**

Every game with **ω-regular** objectives is determined

**Algorithmic questions**

- **Who** is the winner from initial vertex $v_0$?
- **Complexity** class of this decision problem?
- **Can we construct** a winning strategy for the winner?
- **What kind** of winning strategy? Memoryless, finite-memory?
Algorithmic results for one-player games

Classical question in automata theory: **Player 1 wins** iff there exists a play satisfying the objective

- **Reachability** objective: emptiness of automata on finite words
- **Büchi** objective: emptiness of automata on infinite words
Algorithmic results for one-player games

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Finite-memory winning strategy iff the winning play is eventually periodic
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Finite-memory winning strategy iff the winning play is eventually periodic

- reachable cycle in the graph
- reachable simple cycle for memoryless strategies
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Finite-memory winning strategy iff the winning play is eventually periodic

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- reachable simple cycle for memoryless strategies

Remark: Given a two-player game $G$, by fixing a strategy $\sigma$ for player $p$, we get a one-player (infinite/finite) game $G_\sigma$
Algorithmic results for two-player games

[Bee80, Imm81, EJ91], see also [Zie98, GTW02]

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- Remember the previous examples
- More information on the proofs in the next slides
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Major open problem: can we solve Parity games in $P$?
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Basic model

Extension with **weights** on the edges

---

**Definition**

Two-player zero-sum game $G = (V, V_1, V_2, E, v_0, w)$ as before, with:

- $w : E \rightarrow \mathbb{Z}$ weight function
Basic model

Extension with weights on the edges

Definition

Two-player zero-sum game $G = (V, V_1, V_2, E, v_0, w)$ as before, with:

- $w : E \to \mathbb{Z}$ weight function

Classical payoff $f(\rho)$ of a play $\rho = \rho_0\rho_1\rho_2 \ldots$:

- $\text{Inf}(\rho) = \inf_{n \in \mathbb{N}} w(\rho_n, \rho_{n+1})$
- $\text{LimInf}(\rho) = \lim \inf_{n \to \infty} w(\rho_n, \rho_{n+1})$
- Total-payoff $\text{TP}(\rho) = \lim \inf_{n \to \infty} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$
- Mean-payoff $\text{MP}(\rho) = \lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$
Basic model

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- Mean-payoff $\overline{\text{MP}}(\rho) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})$

Similar definitions with $\text{Sup}(\rho), \text{LimSup}(\rho), \overline{\text{TP}}(\rho), \overline{\text{MP}}(\rho)$
Quantitative objectives

Example

\[ \text{play } \rho = (v_0 v_1)^\omega \]

\[ \text{TP}(\rho) = \lim \inf_{n \to \infty} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1}) \]
\[ (-1, -1, -2, -2, -3, -3, \ldots, -n, -n, \ldots) \to -\infty = \text{TP}(\rho) = \overline{\text{TP}}(\rho) \]

\[ \text{MP}(\rho) = \lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1}) \]
\[ (-\frac{1}{1}, -\frac{1}{2}, -\frac{2}{3}, -\frac{2}{4}, -\frac{3}{5}, -\frac{3}{6}, \ldots, -\frac{n}{2n-1}, -\frac{n}{2n}, \ldots) \to -\frac{1}{2} = \text{MP}(\rho) = \overline{\text{MP}}(\rho) \]
Quantitative objectives

Example

\[\text{play } \rho = (v_0 v_1)^\omega\]

- \[\text{TP}(\rho) = \lim\inf_{n \to \infty} \sum_{k=0}^{n-1} w(\rho_k, \rho_{k+1})\]
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**Lemma:** If eventually periodic \( \rho = hg^\omega \), then \( \text{MP}(\rho) = \overline{\text{MP}}(\rho) = \) mean payoff of the cycle \( g \)
Quantitative objectives

Definition

Classical quantitative objectives are, given a threshold $\nu \in \mathbb{Q}$:

- **Inf** objective: ensure $\text{Inf}(\rho) \geq \nu$
- Similarly for the other payoff functions **LimInf**, **TP**, ...
Quantitative objectives

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- Similarly for the other payoff functions \( \text{LimInf} \), \( \text{TP} \), ...

- Player 1 wants to maximize the payoff (interest in the highest \( \nu \))
- Player 2 wants to minimize the payoff
Quantitative objectives

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- Similarly for the other payoff functions \( \text{LimInf}, \text{TP}, \ldots \)

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Corollary of Martin’s Theorem

Games with such quantitative objectives are determined

- \( \text{Inf}, \text{LimInf} \): \( \omega \)-regular
- \( \text{TP}, \text{MP} \): not \( \omega \)-regular, but Borel
Algorithmic results for two-player games

[EM79, ZP96, BSV04]

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Reductions:
- Parity games $\rightarrow$ Mean-payoff games [Jur98]
- Total-payoff games $\leftrightarrow$ Mean-payoff games

Major open problem: can we solve Parity, Mean-payoff and Total-payoff games in $P$?
Algorithmic results

**Theorem [GZ04]**

Let $G$ be a weighted game. If the payoff function $f$ is fairly mixing, i.e.:

1. $f(\rho) \leq f(\rho') \Rightarrow f(h\rho) \leq f(h\rho')$
2. $\min\{f(\rho), f(h^\omega)\} \leq f(h\rho) \leq \max\{f(\rho), f(h^\omega)\}$
3. $\min\{f(h_0 h_2 h_4 \ldots), f(h_1 h_3 h_5 \ldots), \inf_i f(h_i^\omega)\} \leq f(h_0 h_1 h_2 h_3 \ldots) \leq \max\{f(h_0 h_2 h_4 \ldots), f(h_1 h_3 h_5 \ldots), \sup_i f(h_i^\omega)\}$

then both players have memoryless (optimal) winning strategies.
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then both players have **memoryless** (optimal) winning strategies

- **Many applications**: Parity, Inf, LimInf, Mean-payoff, Total-payoff, ...  
- If the payoff function is prefix-independent, i.e. $f(\rho) = f(h\rho)$, then conditions 1. and 2. are satisfied
- **Simple proof** by induction on the number of edges
Algorithmic results

Parity games in $\text{NP} \cap \text{co-NP}$
Algorithmic results

Parity games in $\text{NP} \cap \text{co-NP}$

- in $\text{NP}$:
  - **Guess** a memoryless winning strategy $\sigma$ player 1
  - In the one-player game $G_\sigma$, check in polynomial time whether there exists a reachable cycle with odd maximum color
Algorithmic results

Parity games in $\mathsf{NP} \cap \mathsf{co-NP}$

- **in NP:**
  - Guess a memoryless winning strategy $\sigma$ player 1
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- **in co-NP:** symmetrically for player 2
Algorithmic results

Parity games in $NP \cap co-NP$

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- in $co-NP$: symmetrically for player 2

Mean-payoff games in $NP \cap co-NP$

- Same approach
  - One can compute in polynomial time the minimum (resp. maximum) mean weight cycle in a weighted graph [Kar78]
Algorithmic results

Parity games in NP \( \cap \) co-NP

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Mean-payoff games in NP \( \cap \) co-NP

- Same approach

- One can compute in polynomial time the minimum (resp. maximum) mean weight cycle in a weighted graph [Kar78]
Window games

Objective to satisfy inside a finite window sliding along the play [CHH09], [CDRR15, BHRR16]

Definition

Given a window size $\lambda$ and a threshold $\nu$,

- WMP objective: ensure for all positions $i$ of the window,

$$\frac{1}{\ell} \sum_{k=0}^{\ell-1} w(\rho_{i+k}, \rho_{i+k+1}) \geq \nu$$

for some $\ell \leq \lambda$
Window games

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Definition

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  \]

Given a window size \( \lambda \) and a coloring \( c \)

- **WParity** objective: ensure for all positions \( i \) of the window,
  \[
  \max_{0 \leq k \leq \ell} c(\rho_{i+k}) \text{ even for some } \ell \leq \lambda
  \]
Window games

Motivations

- **Strengthening** of the Mean-payoff and Parity objectives
  - Guarantee within a bounded time, not in the limit
- More **computationally** tractable
  - (Open problem: Mean-payoff and Parity games are in P?)
Window games

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**Remark:** $\omega$-regular objectives, thus determined games
Window games

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Remark: $\omega$-regular objectives, thus determined games

Example

- Player 1 winning for Parity
- but loosing for WParity for all window sizes $\lambda$
Window games

Results

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Model

Extension to \textit{k-dimensional} weighted games with \(k \geq 2\)

Definition

Two-player zero-sum game \(G = (V, V_1, V_2, E, v_0, w)\) as before except:

\(w : E \rightarrow \mathbb{Z}^k\) weight function
Model

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- \( w : E \rightarrow \mathbb{Z}^k \) weight function

Definition

Quantitative objective \( \Omega \):

- Intersection \( \bigcap_{\ell=1}^{k} \Omega_\ell \) such that \( \Omega_\ell \) is an objective for dimension \( \ell \)
- More generally, Boolean combination of such objectives \( \Omega_\ell \), \( 1 \leq \ell \leq k \)
Example

- One-player game
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  with \( \Omega_1 = \text{MP}(\rho) \geq 1 \) for dimension 1
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    with $a + b + c = 1$ and $b > 0$
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- Player 1 is **winning** with **infinite-memory** strategies
  - Alternate visits to $v_0$ and $v_1$
  - At alternation $n$, loop $n$ times on $v_0$ and then loop $n$ times on $v_1$
    $$a_n \cdot (2, 0) + \epsilon_n \cdot (0, 0) + a_n \cdot (0, 2)$$
    with $\epsilon_n \to 0$ and $a_n \to \frac{1}{2}$
  - Mean-payoff equal to $(1, 1)$
## Results with the same objective on all dimensions

[CDHR10], [CRR14, CDERR15]

<table>
<thead>
<tr>
<th>Complexity</th>
<th>TP, TP</th>
<th>MP</th>
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<tbody>
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[CDRR15, BHRR16]

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Results with heterogeneous objectives

Only preliminary results

Theorem

- [Vel15]: Undecidability for Boolean combinations of $\text{MP}$ and $\overline{\text{MP}}$ objectives
- [BHR15]: Decidability for Boolean combinations of WMP, Inf, Sup, LimInf, LimSup objectives, and finite-memory winning strategies for both players
<table>
<thead>
<tr>
<th>Topic</th>
<th>2-player games</th>
<th>Quantitative 2-player games</th>
<th>$k$-dim 2-player games</th>
<th>Multiplayer games</th>
<th>Conclusion</th>
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Model

Summary

- 2-player zero-sum games, one player against the other
- Qualitative/quantitative uni/multidimensional objectives
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Multiplier non zero-sum games

- Several players with their own objective
- Non necessarily antagonistic objectives
Model

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- Non necessarily antagonistic objectives

Definition

\[ n \text{-player non zero-sum game } G = (V, (V_p)_{p \in \Pi}, E, v_0, \bar{w}): \]

- Set \( \Pi \) of \( n \) players, \( n \geq 1 \)
- \( (V_p)_{p \in \Pi} \) partition of \( V \) with \( V_p \) controlled by player \( p \in \Pi \)
- optional: \( \bar{w} = (w_p)_{p \in \Pi} : E \to \mathbb{Z}^n \) such that
  - \( w_p \) is the weight function of player \( p \)
  - leading to his payoff function \( f_p \)
Model

Definition

Objective $\Omega_p$ for each player $p \in \Pi$
- qualitative
- quantitative (depending on $f_p$)

Strategy profile $\bar{\sigma} = (\sigma_p)_{p \in \Pi}$ and outcome $\rho = \langle \bar{\sigma} \rangle_{v_0}$ from initial vertex $v_0$. 

Model

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Objective $\Omega_p$ for each player $p \in \Pi$
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Strategy profile $\bar{\sigma} = (\sigma_p)_{p \in \Pi}$ and outcome $\rho = \langle \bar{\sigma} \rangle_{v_0}$ from initial vertex $v_0$.

Example
- 3-player game (player 1 ⃝, player 2 □, player 3 ♦)
- Each player wants to visit one of his vertices infinitely often
- Outcome $(v_0 v_1 v_2 v_1)^\omega$: good solution for each player that needs cooperation
Nash equilibria (NE)

Classical notion such that each player is
- rational (he wants to maximize his payoff)
- selfish (he is only concerned with his own payoff)
Nash equilibria (NE)

Classical notion such that each player is

- rational (he wants to maximize his payoff)
- selfish (he is only concerned with his own payoff)

Informally, \( \bar{\sigma} \) is an NE if no player has an incentive to deviate from his strategy, if the other players stick to their own strategies.

Definition [Nas50]

The strategy profile \( \bar{\sigma} \) is a Nash equilibrium from \( v_0 \) if, for each player \( p \in \Pi \), for each strategy \( \sigma'_p \) of \( p \),

\[
 f_p(\langle \sigma'_p, \bar{\sigma}_{-p} \rangle_{v_0}) \leq f_p(\langle \bar{\sigma} \rangle_{v_0}).
\]

Notation: \( \bar{\sigma}_{-p} = (\sigma_i)_{i \in \Pi \setminus \{p\}} \).
Nash equilibria

Example of NE: outcome $v_0 v_2 v_4^\omega$ with payoff $(3, 2)$

No incentive to deviate:

- If player 1 deviates to $v_1$, he will get 1 instead of 3
- If player 2 deviates to $v_3$, he will get 1 instead of 2
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Algorithmic questions

- Does there exist an NE from initial vertex $v_0$?
- Can we construct it?
- With what kind of strategies? Memoryless, finite-memory?
Some results

Theorem

Qualitative objectives

- [GU08]: Existence of an NE in case of Borel objectives
Some results

Theorem

Qualitative objectives

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Quantitative objectives

- [Kuh53]: Construction of an NE for games played on a finite tree
- [FL83, Har85]: Existence of an NE if the payoff function $f_p$ of each player $p$ is bounded and continuous
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Proof of [Kuh53]: Backward induction from the leaves to the root
### Some results

#### Theorem

**Qualitative objectives**
- [GU08]: Existence of an NE in case of **Borel objectives**

**Quantitative objectives**
- [Kuh53]: **Construction** of an NE for games played on a **finite tree**
- [FL83, Har85]: Existence of an NE if the payoff function $f_p$ of each player $p$ is **bounded and continuous**

#### Proof of [FL83, Har85]:

- **On the unravelling of the game $G$ truncated at depth $d$, construction of an NE $\bar{\sigma}^d$ by [Kuh53]**
- **A subsequence of $(\bar{\sigma}^d)_{d \in \mathbb{N}}$ converges to a strategy profile $\bar{\sigma}^*$ that is proved to be an NE**
Some results

**Theorem**

**Qualitative objectives**
- [GU08]: Existence of an NE in case of Borel objectives

**Quantitative objectives**
- [Kuh53]: Construction of an NE for games played on a finite tree
- [FL83, Har85]: Existence of an NE if the payoff function $f_p$ of each player $p$ is **bounded** and **continuous**

**Remark:** The LimInf, LimSup and MP payoff functions are not continuous

For example, $\rho^n = v_0^n v_1^\omega \rightarrow \rho = v_0^\omega$ and

$$\lim_{n \rightarrow \infty} \text{LimInf}(\rho^n) = 1 \neq 0 = \text{LimInf}(\rho)$$
Some results

**Theorem [BDS13]**

Construction of a finite-memory NE in the game $G$ if for all $p$

- the payoff function $f_p$ satisfies: $f_p(\rho) \leq f_p(\rho') \Rightarrow f_p(h\rho) \leq f_p(h\rho')$
- the two-player zero-sum game $G^p$
  - where player 1 is $p$ and player 2 is the coalition of the other players
  - is determined with memoryless optimal winning strategies for both players
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**Proof:**

- Let $\sigma_p$ (resp. $\sigma_{-p}$) be an optimal winning strategy of player $p$ (resp. the coalition) in $G^p$
- Construct the profile $\bar{\sigma}$:
  - play as $\sigma_p$ for each player $p$
    - ($p$ plays selfishly and optimally for his own objective)
  - and as soon as some player $p$ deviates, punish $p$ by playing $\sigma_{-p}$
    - (the coalition plays against $p$’s objective)
Other kinds of equilibria

Subgame perfect equilibrium (SPE) [Sel65]

- takes into account the sequential nature of games played on graphs
- i.e., is an NE from the initial vertex $v_0$, but also after every history $h$ of the game
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- [BBMR15]: Construction of a finite-memory SPE for quantitative reachability games
Other kinds of equilibria

Secure equilibrium (SE) [CHJ06]

- each player wants to maximize his payoff, as a *first* objective
- and then minimize the payoff of the other players, as a *second* objective
Other kinds of equilibria

Secure equilibrium (SE) [CHJ06]

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Theorem

- [CHJ06]: Existence of an SE for 2-player games with Borel qualitative objectives
- [DFK+14]: Existence of an SE
  - if the payoff function $f_p$ of each player $p$ is bounded and continuous
  - or if each $f_p$ is Borel measurable and have finite range
- [BMR14]: Previous result [BDS13] extended to SE for 2-player games
1. Topic

2. Qualitative two-player zero-sum games

3. Quantitative two-player zero-sum games

4. Multidimensional two-player zero-sum games

5. Multiplayer non zero-sum games

6. Conclusion
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- Different notions of equilibria (NE, SPE, SE)
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- Stochastic games
- Imperfect information
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Thank you!


